Q. Can size of perm? Congruent? Isolables? 
   
   - Let $f \in \mathbb{F}_q[x]^*$ be of degree $d$. Then $f$ is computed by size $O(d \log q)$.
   
   - $f = \sum_{i=0}^{d} a_i x^i$
   
   (in $\mathbb{F}_q[x]^*$).
   
   - Size is $O(1)$.
   
   - $\text{deg} f = 0(1)$.
   
   Q: Do better? If no, take a guess why we are stuck.

- Let $\text{perm}$ have size $O(n \cdot 2^n)$.

- If yes, then $P(x)$ is computed.

- If no, then $P(x)$ is complicated.

Q. Do better?

- Use recursion and relate larger cases to smaller cases?

- Use recursion and dynamic programming: store, don't recalculate.

- For $x_{i+1} \rightarrow x_{i+1} - x_i$.

- $O(n \cdot 2^n)$.
Q: Gaussian elimination or algebraic circuit?

Use division to eliminate!

requires "branching" in gaussian elimination
requires "zero testing" to halt early, need division zero

proof: case the cases you write, did by issue

check: entire proof when combined, nuke zero and non zero only

Q: det = a (e - ab)

\[
\begin{vmatrix}
  a & b & c \\
  d & e & f \\
  g & h & i
\end{vmatrix}
= a \begin{vmatrix}
  e & f \\
  h & i
\end{vmatrix} - b \begin{vmatrix}
  d & f \\
  g & i
\end{vmatrix} + c \begin{vmatrix}
  d & e \\
  g & h
\end{vmatrix}
\]

\[
\begin{align*}
\text{det} &= a \left( e - ab - h + gf - ch - di \right) \\
&= a \left( e - ab - h + \frac{1}{a} \left( f - dc \right) \right)
\end{align*}
\]
**Correctness:**
- No empty
- No divider by zero

**Complexity:**
- $\text{poly}(n)$ size check or divide
- $\text{poly}(s, \log(\text{depth}))$ size check for divide

**Note:**
- Dot has dot size $O(n \cdot \log n)$, $n > 0$, $n < 2.373$ FMN expression.

**Depth:**
- The depth of a dot check is the length of the longest input/output path on unbounded fan-in the alloca $\text{poly}(n \cdot \log n)$.

**Example:**
- $x_1, \ldots, x_n \rightarrow x_1, \ldots, x_n \rightarrow \text{depth} \in \Omega(n)$

**Note:**
- Depth is a measure of parallel complexity, as in Fig. 2.

**Note:**
- All unbounded fan-in the alloca $\text{poly}(n \cdot \log n)$.

**Strategy:**
- A gate simulator $2$

**Note:**
- Depth is preserved.

**Note:**
- If copying on our depth $\Rightarrow$
- The alloca variable system preserved in $\text{poly}(n \cdot \log n)$

**Step:**
- $\text{esym}(\text{shift, } \text{rep}) = \text{esym}^{\text{in}, \text{out}} - \text{esym}^{\text{in}, \text{in}} + x_n \cdot \text{esym}^{\text{in}, \text{out}, \text{out}}$

**Conclusion:**
- Clear.

**Complexity:**
- $O(n)$ depth
- $O(\text{work per layer})$
Q: do better?

\[ f: F^n \rightarrow \mathbb{R} \quad \text{linear map}\]

\[ \text{rank } = \text{max } i: \text{ it is invertible } \quad \Rightarrow \text{ is surjective}\]

Consider a set of linearly independent vectors \( \{v_i\}_{i=1}^n \) in \( F^n \):

\[ L(x) = \sum_{i=1}^n (x - a_i) v_i \quad \text{where } L: (v_i - a_i) \leq 1, i=1, \ldots, n \]

Given a basis \( \{v_i\}_{i=1}^n \) for \( F^n \), the rank of the matrix associated with \( L \) is \( n \).

Q: \( O(1) \) depth decision?

\[ \text{rank } = \text{size of decision}\]

- Symmetry

- \( O(1) \) size decision

- Symmetry

\[ |\text{Bu-ori}| - \text{esym} \cdot \text{nd} \cdot \text{size of decision}\]

\[ \text{esym} = |H_{ij} (x_i - x_j)| \]

\[ \text{esym} \cdot \text{nd} \cdot \text{size of decision} \]

\[ \text{rank} \] is non-homogeneous

\[ \text{esym} \cdot \text{nd} \cdot \text{size of decision} \]

- Symmetry

- \( O(1) \) depth decision

- Symmetry

- \( O(1) \) depth decision

- Symmetry

- \( O(1) \) depth decision

- Symmetry

\[ \text{rank} \] is same size as from linear expression

- Smaller known expression for parameters

- Symmetry

- \( O(1) \) depth decision

- Symmetry

- \( O(1) \) depth decision

- Symmetry

- \( O(1) \) depth decision

- Symmetry

- \( O(1) \) depth decision

- Symmetry

\[ \text{esym} \cdot \text{nd} \cdot \text{size of decision} \]

- Symmetry