Spring 2022, CS 586/IE 519: Combinatorial Optimization
Homework 5
Due: Monday, May 2nd, 2022

Instructions and Policy: Each person should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with. Solutions to most of these problems can be found from one source or the other. Try to solve on your own first, and cite your sources if you do use them.

Please write clearly and concisely. Refer to known facts. You should try to convince us that you know the solution, as quickly as possible.

Submit solutions to all three problems.

Problem 1 Let $f : E \to \mathbb{Z}_+$ be an integer-valued monotone submodular function with $f(\emptyset) = 0$. Such a function is called a polymatroid. Recall that the rank function of a matroid $\mathcal{M} = (E, \mathcal{I})$ is a polymatroid with the additional property that $f(e) \leq 1$ for each $e \in E$. Can every polymatroid be understood via matroids? This problem shows that this is indeed the case. If f is a polymatroid then $f(e)$ can be an integer larger than 1. Given f over E construct a new set X where $X = \uplus_e X_e$ where X_e is a set of $f(e)$ elements (X_e and $X_{e'}$ for $e \neq e'$ are disjoint sets). We define a set function r over ground set X as follows. For $U \subseteq X$ let

$$r(U) = \min_{T \subseteq E} (f(T) + |U \setminus \bigcup_{e \in T} X_e|).$$

- Prove that r is the rank function of a matroid over X.
- Prove that for any $T \subseteq E$, $f(T) = r(\cup_{e \in T} X_e)$.

Problem 2 Let $G = (V, E)$ be a graph. For a subset $S \subseteq V$ of nodes, define the *density* of the induced graph $G[S]$ as $|E[S]|/|S|$ where $E[S]$ is the set of edges with both end points in S. The goal is to find the set of nodes that maximizes the density of the induced graph. This is called the densest subgraph problem.

- Prove that the function $f : 2^V \to \mathbb{Z}_+$ where $f(S) = |E(S)|$ is supermodular and monotone. Recall that a set function f is supermodular iff $-f$ is submodular. Alternatively, f is supermodular if $f(A) + f(B) \leq f(A \cup B) + f(A \cap B)$ for all $A, B \subseteq V$.
- Suppose we are given a non-negative supermodular function $f : 2^V \to \mathbb{R}_+$ and want to solve the problem of finding a set of maximum density. Formally we want to find $\max_{S \subseteq V} \frac{f(S)}{|S|}$. Consider the decision problem: given f as a value oracle and a guess $\lambda > 0$, is there a set $S \subseteq V$ such that $f(S)/|S| \geq \lambda$. Show how one can use submodular set function minimization to solve this efficiently. Use this and binary search to solve $\max_{S \subseteq V} \frac{f(S)}{|S|}$.

Problem 3 Submodular functions arise in interesting ways. One "influential" recent example is in the work of Kempe, Kleinberg, and Tardos on influence maximization in social networks that has resulted in a large number of papers written (whether they are influential or not is for history to judge). They made a conjecture on submodularity in a more general model which was shown to be true in a paper by Mossel and Roch. Summarize the models and results of the two papers in about two pages. This is to make you look at an interesting recent example of submodularity.