



Other Solution Concepts and Game Models

CS580
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Most slides are borrowed from Prof. V. Conitzer's presentations.



So far

- Normal-form games

- Multiple rational players, single shot, simultaneous move

- Nash equilibrium

- Existence
 - Computation in two-player games.

Next:

■ Issues with NE

- Multiplicity
- Selection: How players decide/reach any particular NE

■ Possible Solutions

- Dominance: Dominant Strategy equilibria
- Arbitrator/Mediator: Correlated equilibria, Coarse-correlated equilibria
- Communication/Contract: Stackelberg equilibria, Nash bargaining

■ Other Games

- Extensive-form Games, Bayesian Games

Formally: Games and Nash Equilibrium

- N : Set of players/agents

- $i \in N$, S_i : Set of strategies/moves of player i

- $s = (s_1, \dots, s_n) \in S_1 \times S_2 \times \dots \times S_n$, (s_i, s_{-i})
" $s_{-i} = (s_1, s_2, \dots, s_{i-1}, \dots, s_{i+1}, \dots, s_n)$

$u_i(s)$: payoff/utility of player i

- $\sigma_i \in \Delta(S_i)$ randomized strategy of i

- Probability distribution over the moves in S_i

- *Nash equilibrium*: $\sigma = (\sigma_1, \dots, \sigma_n)$ s.t.

$\forall i \in N, \quad u_i(\sigma_i, \sigma_{-i}) \geq u_i(\tau_i, \sigma_{-i}), \quad \forall \tau_i \in \Delta(S_i)$

$(\sigma_1, \sigma_2, \dots, \sigma_{i-1}, \dots, \sigma_{i+1}, \dots, \sigma_n)$

Dominance

- **Strict dominance:** For a player, move s **strictly dominates** t if no matter what others play, s gives her better payoff than t
 - for all s_{-i} , $u_i(s, s_{-i}) > u_i(t, s_{-i})$
- s **weakly dominates** t if
 - for all s_{-i} , $u_i(s, s_{-i}) \geq u_i(t, s_{-i})$; and
 - for some s_{-i} , $u_i(s, s_{-i}) > u_i(t, s_{-i})$

$-i$ = "the player(s) other than i "

strict dominance

weak dominance

	L	M	R
U	0, 0	1, -1	1, -1
G	-1, 1	0, 0	-1, 1
B	-1, 1	1, -1	0, 0

Dominant Strategy Equilibrium

Playing move s is best for me, no matter what others play.

- $s = (s_1, \dots, s_n)$ is DSE if for each player i , there is a (strategy) move s_i that (weakly) dominates all other moves.

□ for all $i, s'_i, s_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i});$

Example?

Prisoner's Dilemma

- Pair of criminals has been caught
- They have two choices: {confess, don't confess}

	confess	don't confess
confess	-5, -5	0, -6
don't confess	-6, 0	-1, -1

“Should I buy an SUV?”

purchasing cost



cost: 5



cost: 3

accident cost

cost: 5



cost: 5

cost: 8



cost: 2

cost: 5



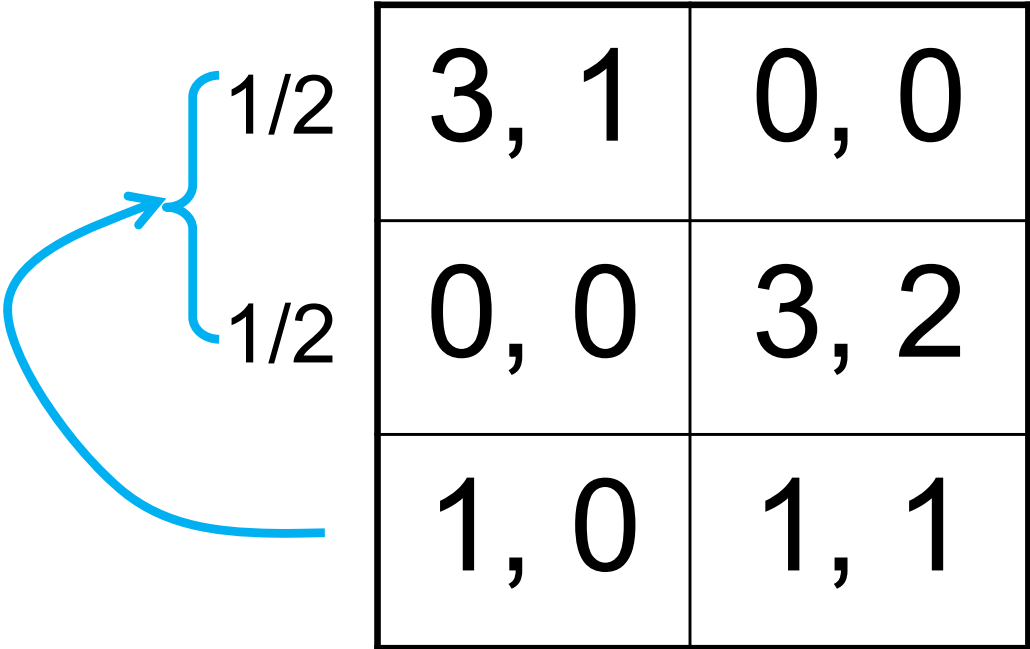
cost: 5



-10, -10	-7, -11
-11, -7	-8, -8

Dominance by Mixed strategies

- Example of dominance by a mixed strategy:

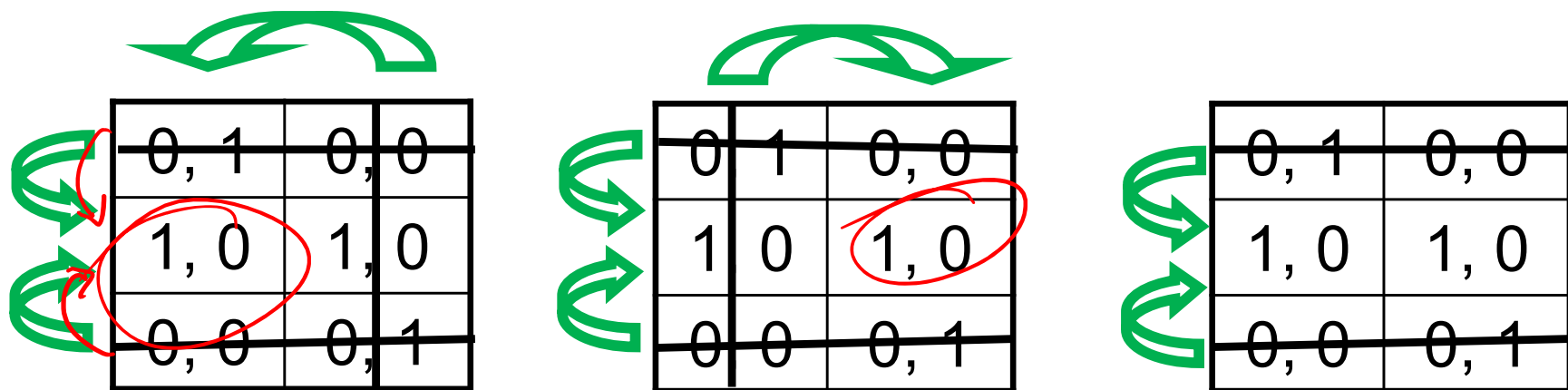


A 3x2 payoff matrix is shown. To the left of the matrix, a blue bracket groups the first two rows, with an arrow pointing to the first column. The bracket is labeled with $1/2$ for the top row and $1/2$ for the bottom row, indicating a mixed strategy.

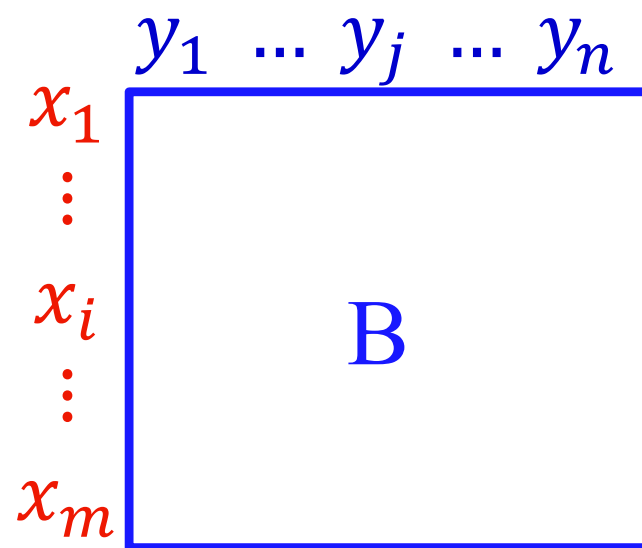
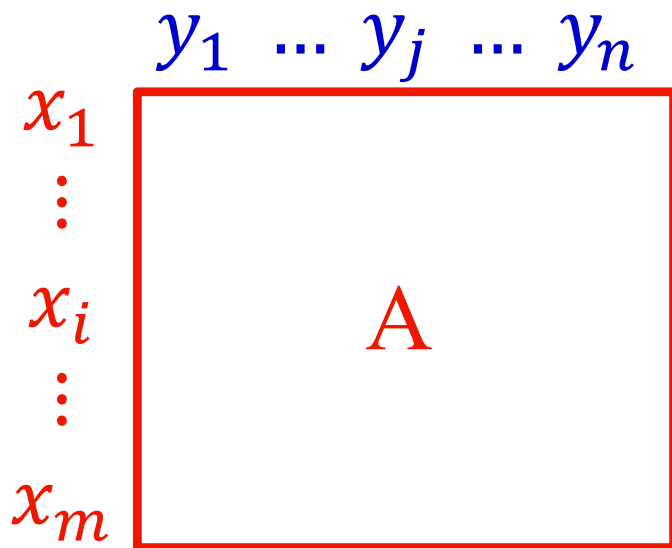
3, 1	0, 0
0, 0	3, 2
1, 0	1, 1

Iterated dominance: path (in)dependence

Iterated **weak dominance** is **path-dependent**: sequence of eliminations may determine which solution we get (if any)
(whether or not dominance by mixed strategies allowed)



Iterated **strict dominance** is **path-independent**: elimination process will always terminate at the same point
(whether or not dominance by mixed strategies allowed)



NE: $x^T A y \geq x'^T A y, \forall x'$ $x^T B y \geq x^T B y', \forall y'$

No one plays
dominated
strategies.

Why?

What if they can discuss beforehand?

No,

Players: {Alice, Bob}

Two options: {Football, Tennis}

		$\frac{2}{3}$		$\frac{1}{3}$	
		F		T	
$\frac{1}{3}$	F	1 2 0.5		0 0	
$\frac{2}{3}$	T	0 0		2 1 0.5	

At Mixed NE
both get $\frac{2}{3} < 1$



Instead they agree on $\frac{1}{2}(F, T), \frac{1}{2}(T, F)$

Payoffs are (1.5, 1.5) Fair!

Needs a common coin toss!

Correlated Equilibrium – (CE)

(Aumann'74)

- **Mediator** declares a joint distribution P over $S = \times_i S_i$
- Tosses a coin, chooses $s = (s_1, \dots, s_n) \sim P$.
- Suggests s_i to player i **in private**
- P is at **equilibrium** if each player wants to follow the **suggestion** when others do.
 - $U_i(s_i, P_{(s_i, \cdot)}) \geq U_i(s'_i, P_{(s_i, \cdot)}), \forall s'_i \in S_i$

CE for 2-Player Case

- **Mediator** declares a joint distribution P
- Tosses a coin, chooses $(i, j) \sim P$.
- Suggests i to Alice, j to Bob, in private.

$$P = \begin{bmatrix} p_{11} & \dots & p_{1n} \\ p_{i1} & \dots & p_{in} \\ p_{m1} & \dots & p_{mn} \end{bmatrix}$$

$i \rightarrow$ (pointing to the row index) $j \downarrow$ (pointing to the column index)

$$\sum_j A_{ij} P_{ij} \geq \sum_j A_{i'j} P_{ij}$$

- P is a CE if each player wants to follow the suggestion, when the other does.

Given Alice is suggested i , she knows Bob is suggested $j \sim P(i, \cdot)$

$$\langle A(i, \cdot), P(i, \cdot) \rangle \geq \langle A(i', \cdot), P(i, \cdot) \rangle \quad : \forall i' \in S_1$$

$$\langle B(\cdot, j), P(\cdot, j) \rangle \geq \langle B(\cdot, j'), P(\cdot, j) \rangle \quad : \forall j' \in S_2$$

Players: {Alice, Bob}

Two options: {Football, Shopping}

	F	S
F	1 2 0.5	0 0
S	0 0	2 1 0.5

Instead they agree on $\frac{1}{2}(F, S), \frac{1}{2}(S, F)$

Payoffs are (1.5, 1.5) Fair!

CE!

Prisoner's Dilemma

	C	NC
C	-5, -5 0 1	0, -6 0 0
NC	-6, 0 0 0	-1, -1 1 0

C strictly dominates NC

Rock-Paper-Scissors (Aumann)

	R	P	S
R	0, 0 0	0, 1 1/6	1, 0 1/6
P	1, 0 1/6	0, 0 0	0, 1 1/6
S	0, 1 1/6	1, 0 1/6	0, 0 0

$$\left(\frac{1}{2}, \frac{1}{2}, 0\right) = \frac{\frac{1}{6} + \frac{1}{6} + 0}{\frac{1}{6} + \frac{1}{6} + 0}$$

When Alice is suggested R

Bob must be following $P_{(R, \cdot)} \sim (0, 1/6, 1/6)$

Following the suggestion gives her 1/6

While P gives 0, and S gives 1/6.

Computation: Linear Feasibility Problem

Game (A, B). Find, joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$

$$\begin{aligned} \frac{1}{\sum_j p_{ij}} \sum_j A_{ij} p_{ij} &\geq \frac{1}{\sum_j p_{i'j}} \sum_j A_{i'j} p_{ij} \quad \forall i, i' \in S_1 \\ \frac{1}{\sum_i p_{ij}} \sum_i B_{ij} p_{ij} &\geq \frac{1}{\sum_i p_{ij'}} \sum_i B_{ij'} p_{ij} \quad \forall j, j' \in S_2 \\ \sum_{ij} p_{ij} &= 1; \quad p_{ij} \geq 0, \quad \forall (i, j) \end{aligned}$$

Computation: Linear Feasibility Problem

Game (A, B). Find, joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$

$$\sum_j A_{ij} p_{ij} \geq \sum_j A_{i'j} p_{ij} \quad \forall i, i' \in S_1$$

$$\sum_i B_{ij} p_{ij} \geq \sum_i B_{ij'} p_{ij} \quad \forall j, j' \in S_2$$

$$\sum_{ij} p_{ij} = 1; \quad p_{ij} \geq 0, \quad \forall (i, j)$$

N-player game: Find distribution P over $S = \times_{i=1}^N S_i$

$$\text{s.t. } U_i(s_i, P_{(s_i, \cdot)}) \geq U_i(s'_i, P_{(s_i, \cdot)}), \quad \forall s_i, s'_i \in S_i$$

$$\uparrow \sum_{s \in S} P(s) = 1$$

$$\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) P(s_i, s_{-i}) \quad \text{Linear in } P \text{ variables!}$$

Computation: Linear Feasibility Problem

N-player game: Find distribution P over $S = \times_{i=1}^N S_i$

$$\text{s.t. } U_i(s_i, P_{(i,.)}) \geq U_i(s'_i, P_{(s_i,.)}), \quad \forall s_i, s'_i \in S_i$$

$$\uparrow \sum_{s \in S} P(s) = 1$$

$$\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) P(s_i, s_{-i}) \quad \text{Linear in } P \text{ variables!}$$

Can optimize any convex function as well!

Coarse-Correlated Equilibrium

- After mediator declares P , each player opts in or out.
- Mediator tosses a coin, and chooses $s \sim P$.
- If player i opted in, then the mediator suggests her s_i in private, and she has to obey.
- If she opted out, then (knowing nothing about s) plays a fixed strategy $t \in S_i$
- At equilibrium, each player wants to opt in, if others are opting in.

$$U_i(P) \geq U_i(t, P_{-i}), \quad \forall t \in S_i$$

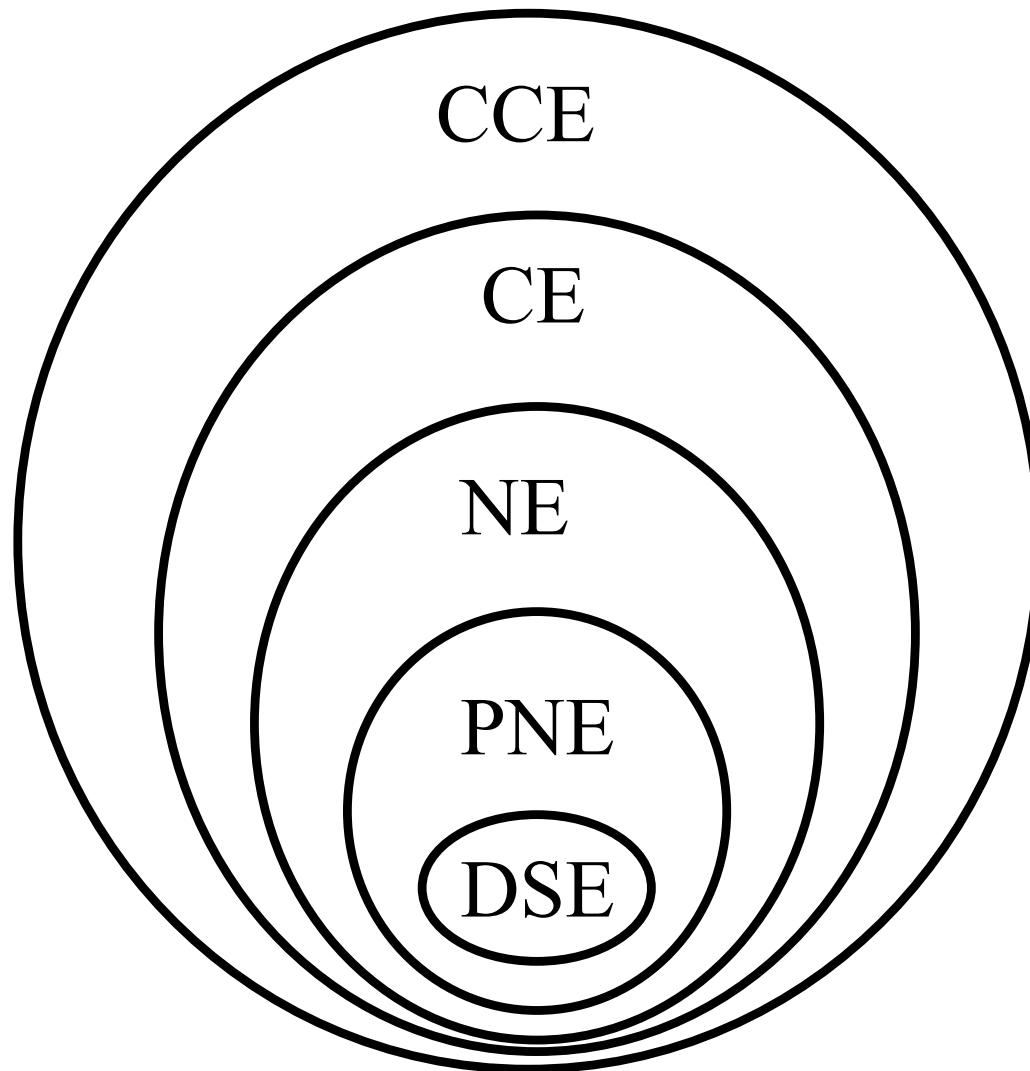
Where P_{-i} is joint distribution of all players except i .



Importance of (Coarse) CE

- Natural dynamics quickly arrive at approximation of such equilibria.
 - No-regret, Multiplicative Weight Update (MWU)
- Poly-time computable in the size of the game.
 - Can optimize a convex function too.

Show the following



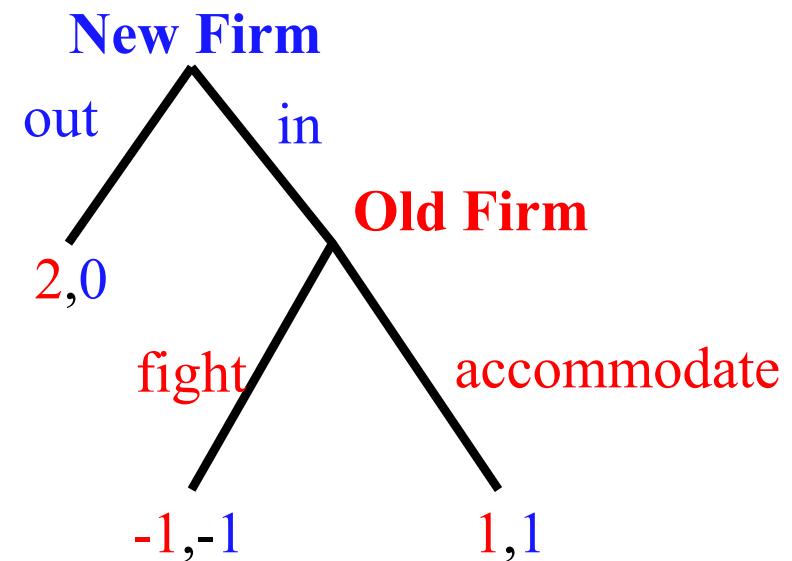
Extensive-form Game

- Players move one after another

- ☐ Chess, Poker, etc.
- ☐ Tree representation.

Strategy of a player:
What to play at each of its node.

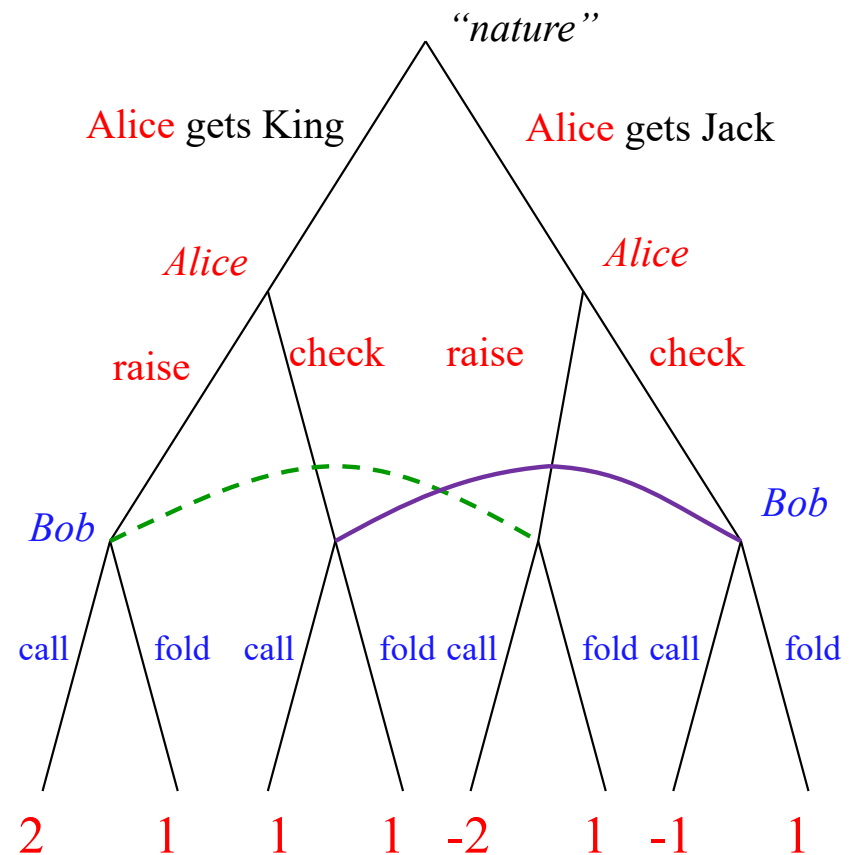
	I	O
F	-1, -1	2, 0
A	1, 1	2, 0



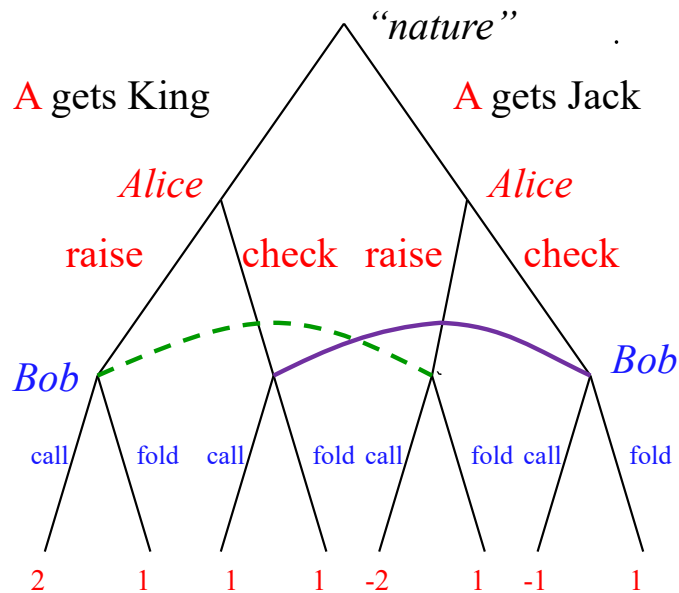
Entry game

A poker-like game

- Both players put 1 chip in the pot
- **Alice** gets a card (King is a winning card, Jack a losing card)
- **Alice** decides to raise (add one to the pot) or check
- **Bob** decides to call
(match) or fold (**Alice** wins)
- If **Bob** called, **Alice**'s
card determines
pot winner



Poker-like game in normal form

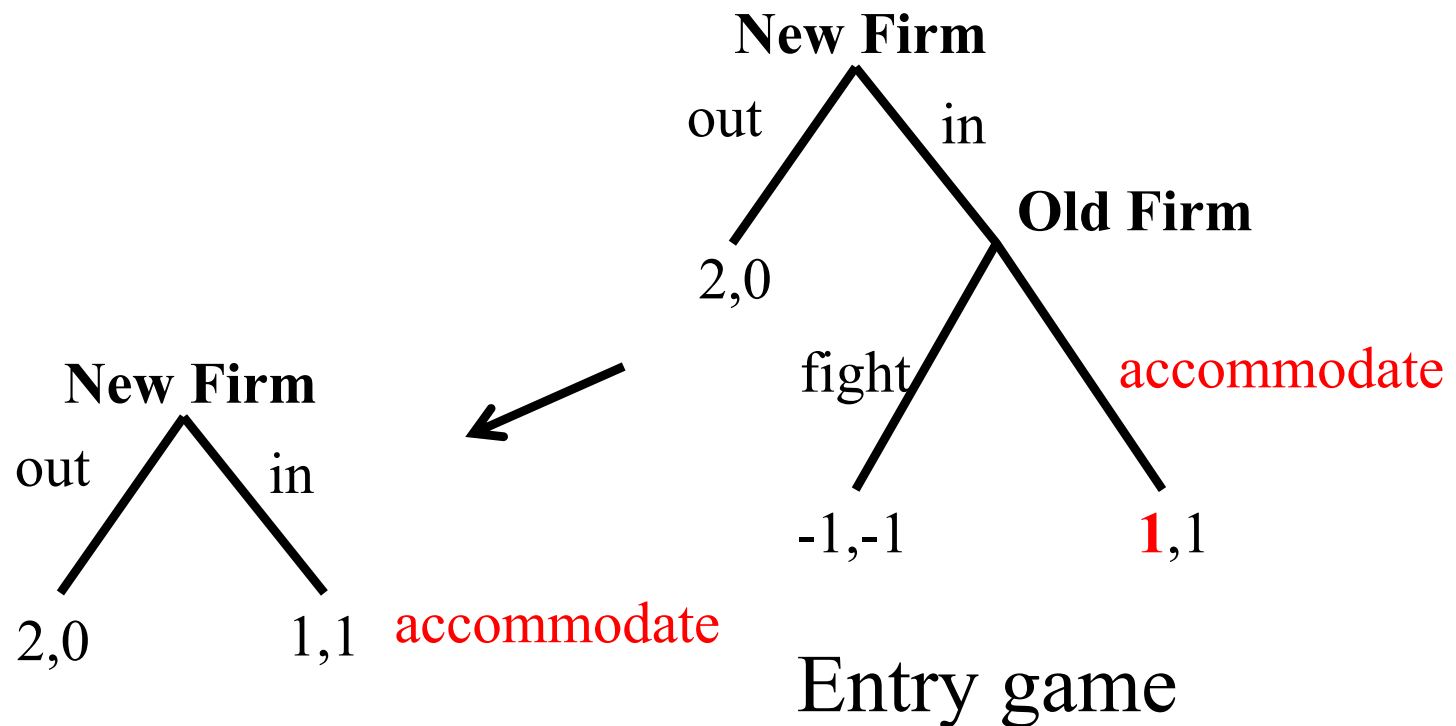


	cc	cf	fc	ff
rr	0, 0	0, 0	1, -1	1, -1
rc	.5, -.5	1.5, -1.5	0, 0	1, -1
cr	-.5, .5	-.5, .5	1, -1	1, -1
cc	0, 0	1, -1	0, 0	1, -1

Can be exponentially big!

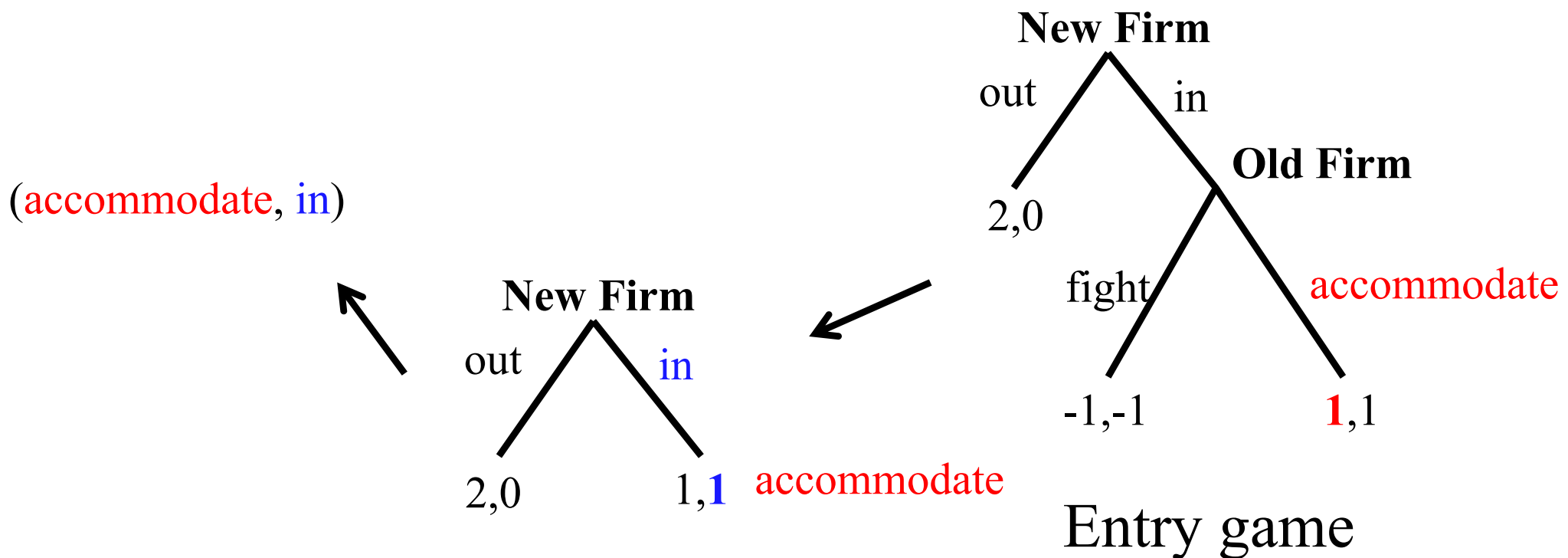
Sub-Game Perfect Equilibrium

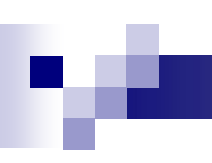
- Every sub-tree is at equilibrium
- Computation when perfect information (no nature/chance move): **Backward induction**



Sub-Game Perfect Equilibrium

- Every sub-tree is at equilibrium
- Computation when perfect information (no nature/chance move): **Backward induction**





Corr. Eq. in Extensive form Game

- How to define?
 - CE in its normal-form representation.
- Is it computable?
 - Recall: exponential blow up in size.
- Can there be other notions?

See “Extensive-Form Correlated Equilibrium: Definition and Computational Complexity” by von Stengel and Forges, 2008.



Security Games

Bargaining

Meanfield Games