



Games and Nash Equilibrium

CS 580

Instructor: [Ruta Mehta](#)



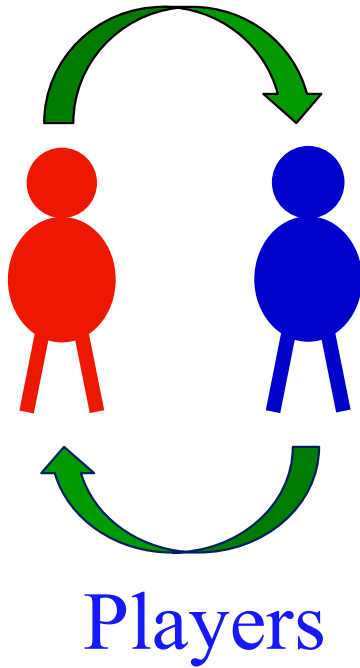
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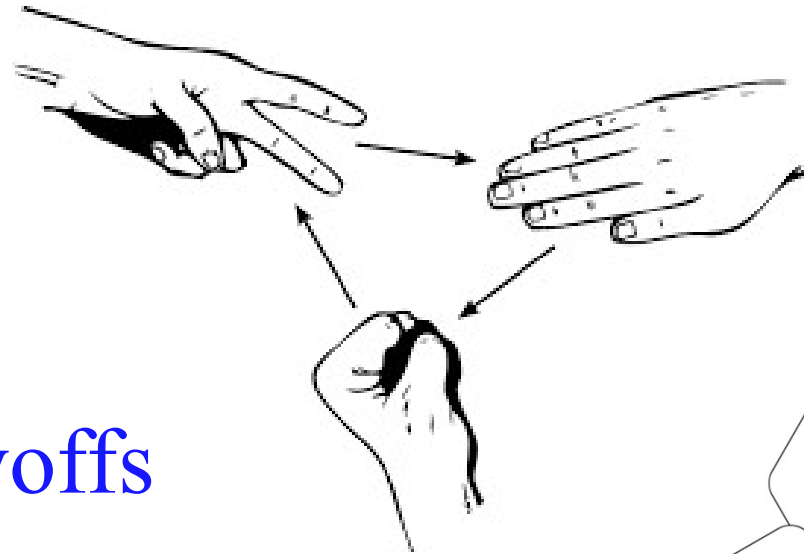
Agenda

- Two-player Games
- Nash Equilibrium (NE)
 - NE existence
 - NE characterization
- Zero-sum games
 - Minmax Theorem
 - LP-duality

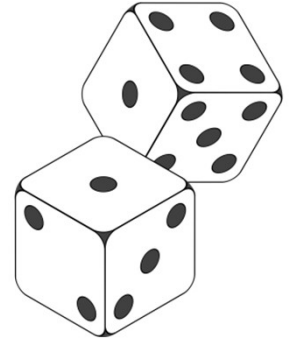
Games



Payoffs

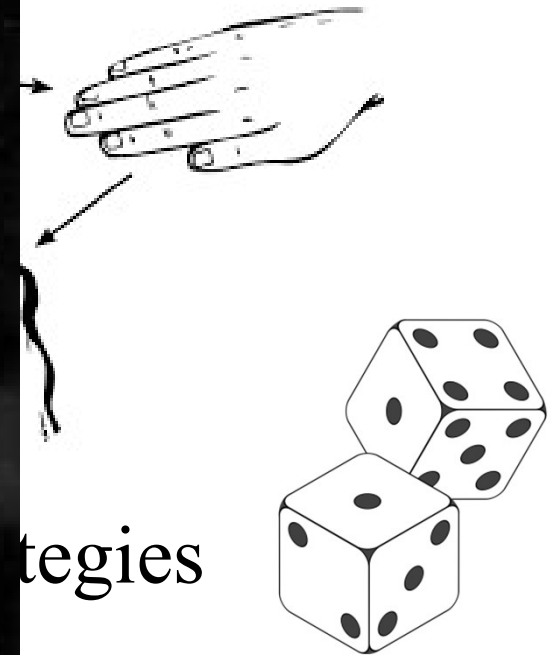
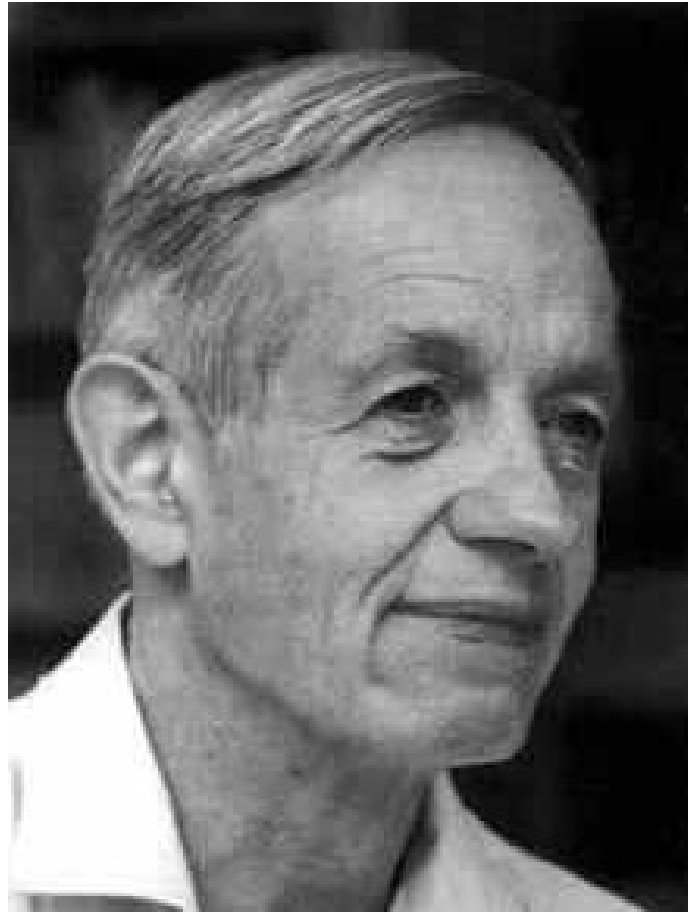
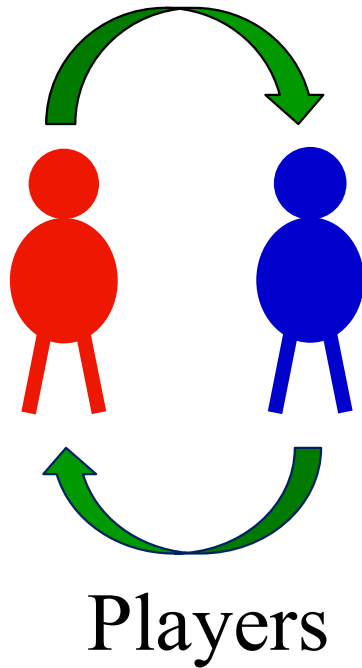


Strategies



Randomize!

Games



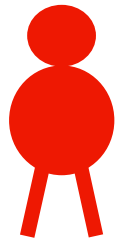
Randomize!

Nash (1950):

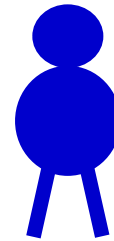
There exists a (stable) state where no player gains by unilateral deviation.

Nash equilibrium (NE)

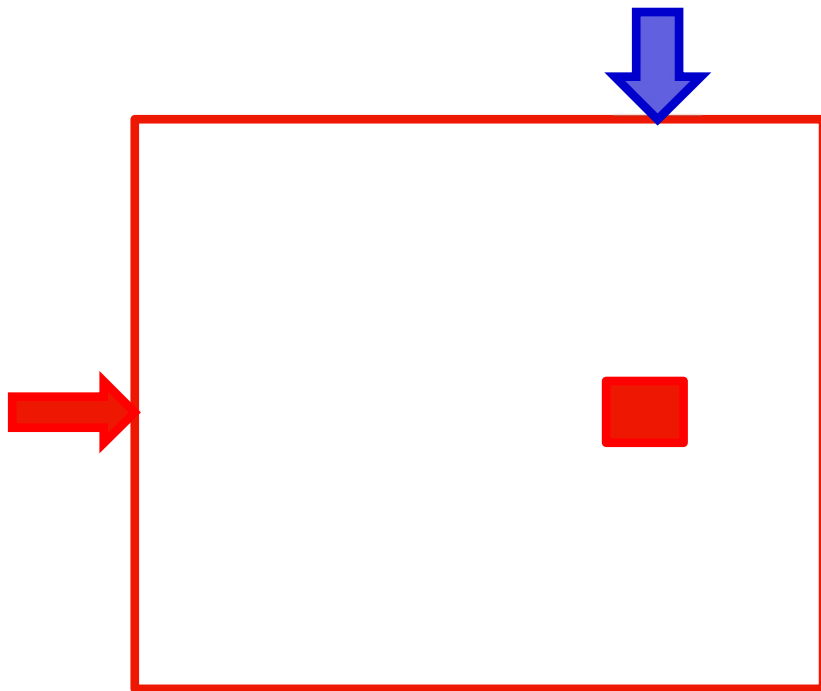
Our focus: Two-player games



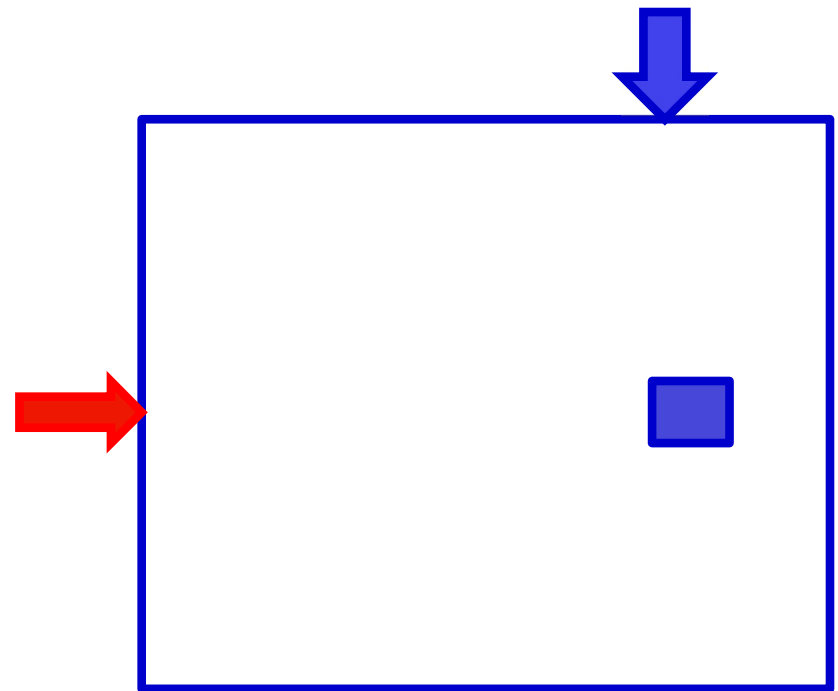
Alice
m strategies



Bob
n strategies



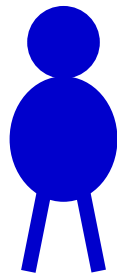
$A_{m \times n}$



$B_{m \times n}$

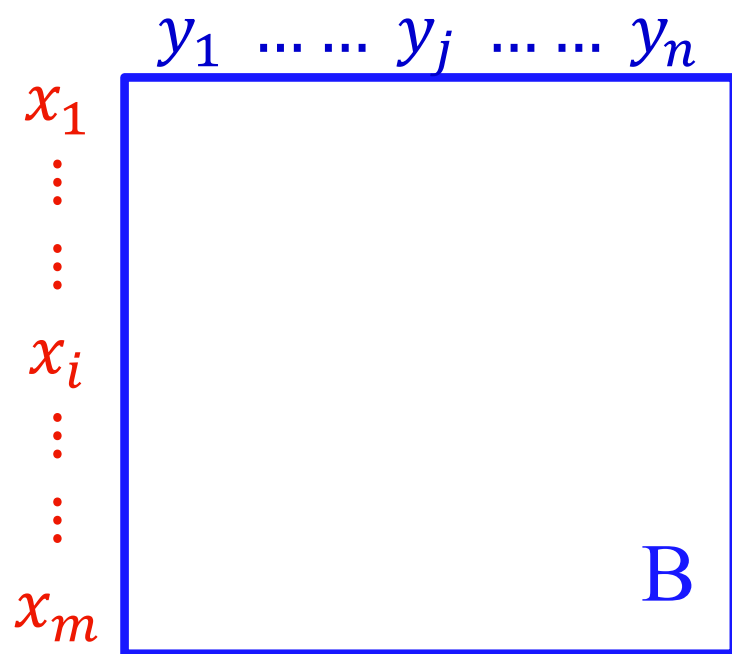
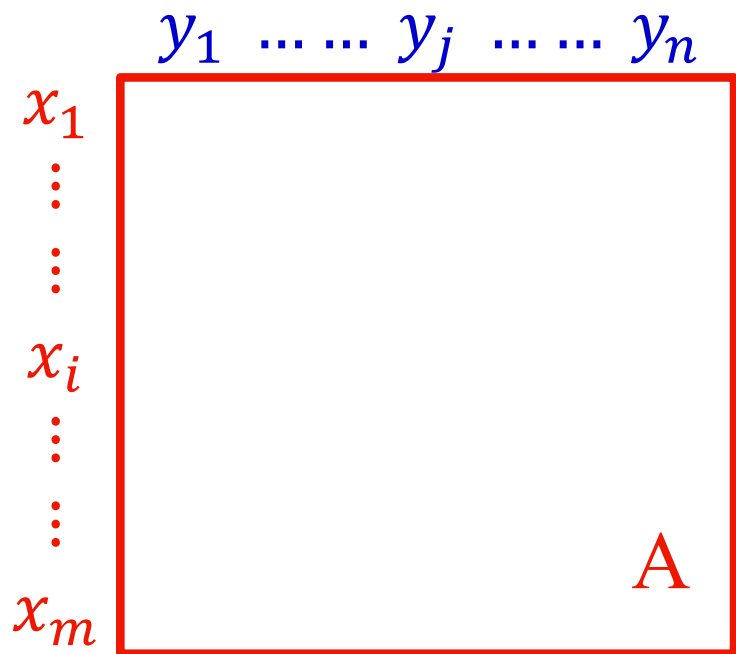


Alice



Bob

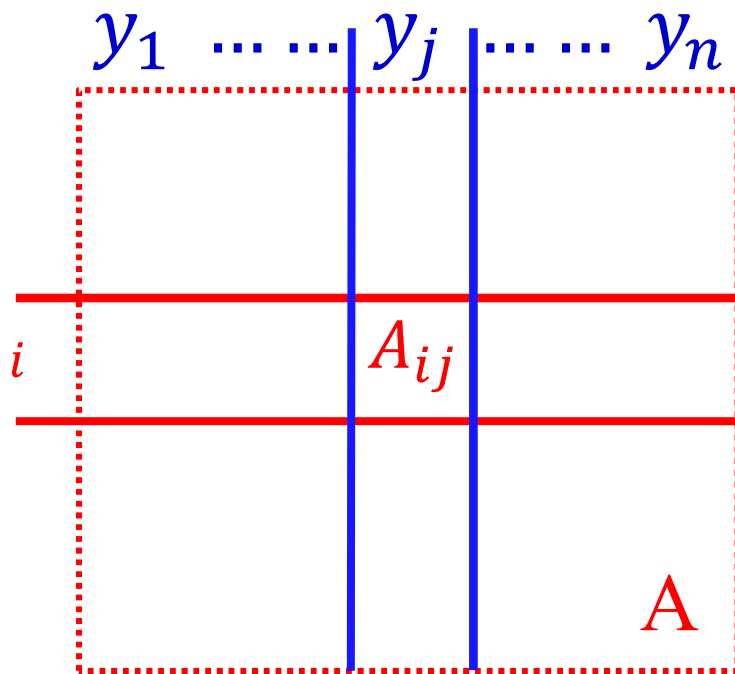
Randomize



2-Nash Characterization



- For **Alice**, i^{th} strategy gives

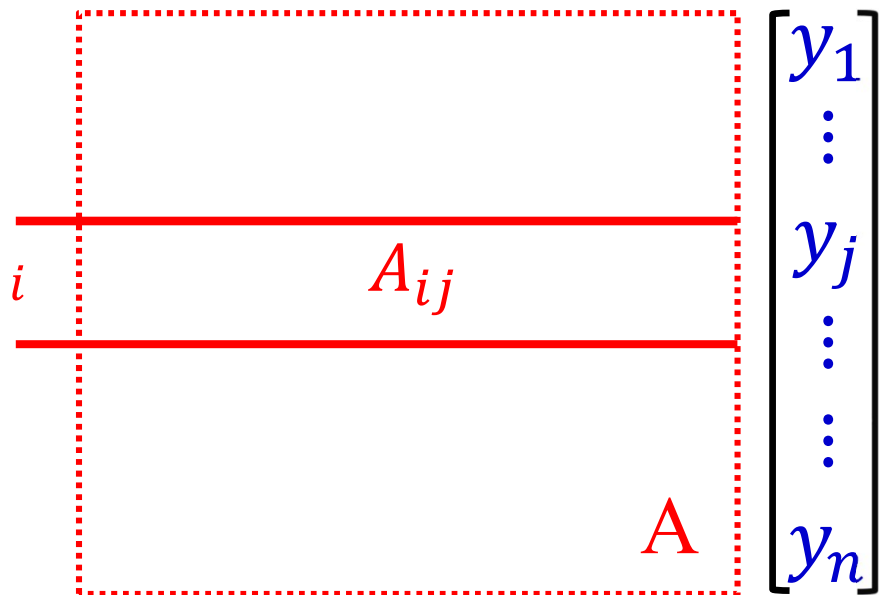


$$\longrightarrow \sum_j A_{ij} y_j$$

2-Nash Characterization



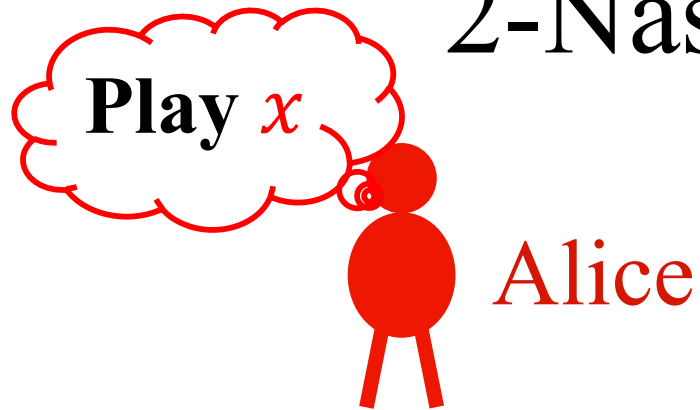
- For **Alice**, i^{th} strategy gives



The diagram illustrates the calculation of the i^{th} component of the product Ay . It shows a red dotted rectangle representing a row of the matrix A . The row is labeled with i on the left and A_{ij} in the center. To the right of the rectangle is a column vector y enclosed in black brackets, with elements $y_1, \dots, y_j, \dots, y_n$ listed vertically. An arrow points from the diagram to the equation $\sum_j A_{ij} y_j = (Ay)_i$.

$$\sum_j A_{ij} y_j = (Ay)_i$$

2-Nash Characterization

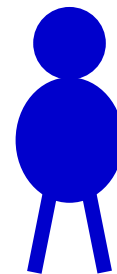


- Alice's expected payoff is

The diagram illustrates the calculation of Alice's expected payoff. On the left, a matrix A is shown with rows indexed by $x_1, \dots, x_i, \dots, x_m$ and columns indexed by $y_1, \dots, y_j, \dots, y_n$. The element A_{ij} is highlighted at the intersection of row i and column j . This matrix is multiplied by a column vector y (with elements $y_1, \dots, y_j, \dots, y_n$). An arrow points to the resulting dot product formula:
$$\sum_i x_i (Ay)_i = x^T Ay$$

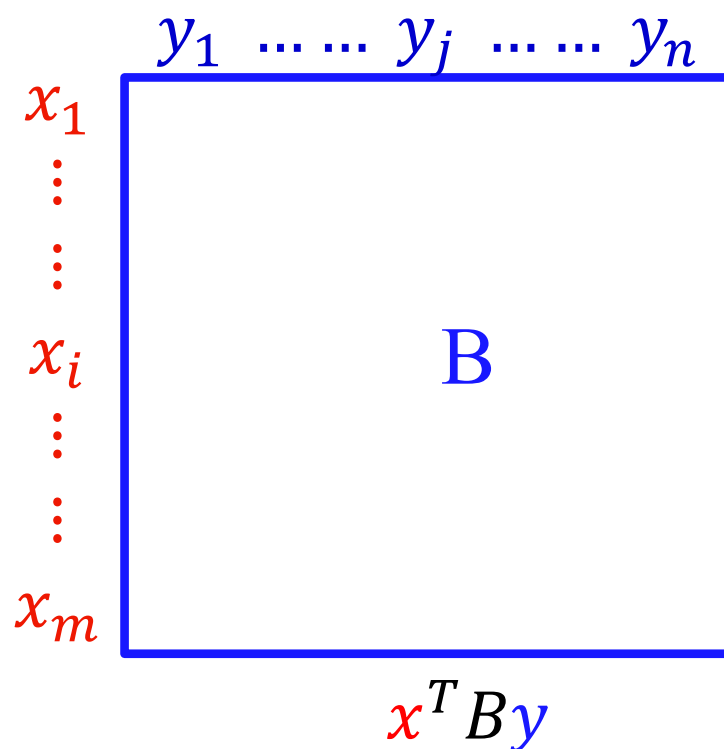
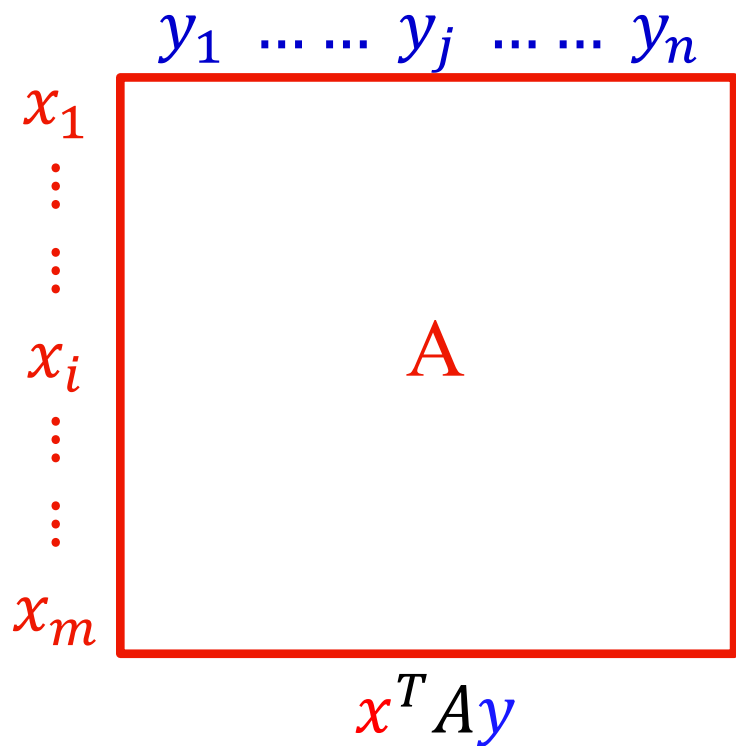


Alice



Bob

Randomize



NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \quad \forall z \in \Delta_m$$

$$x^T B y \geq x^T B z, \quad \forall z \in \Delta_n$$

Example: Matching Pennies

	H	T
H	1 -1	-2 2
T	-2 2	1 -1

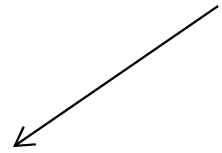


Nash's Existence Theorem (1950)

Brouwer's Fixed Point Theorem

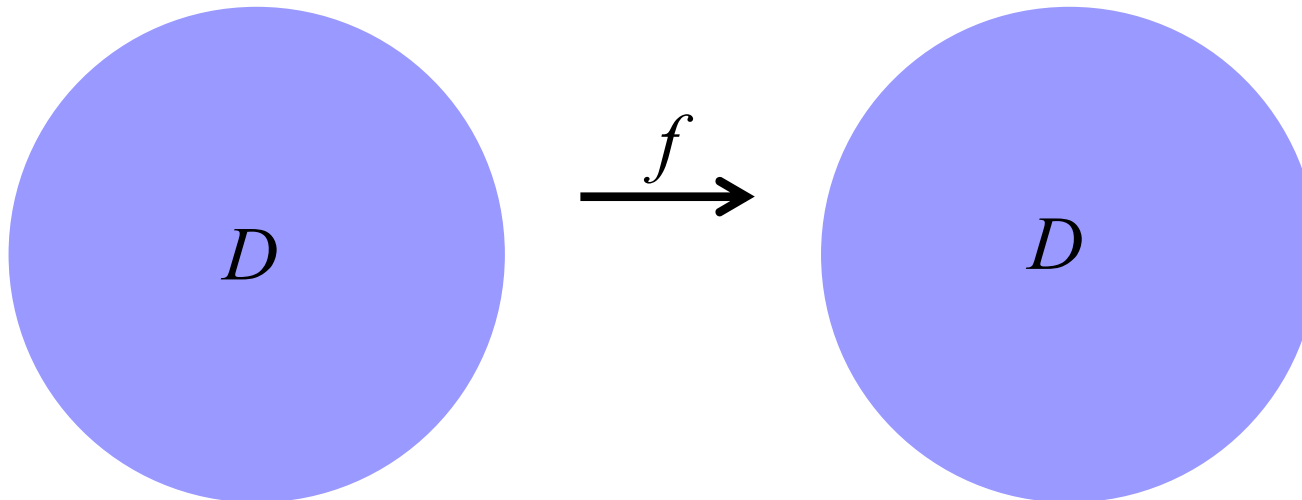
[Brouwer 1910]: Let $f: D \rightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

Then there exists an $x \in D$ s.t. $x = f(x)$.

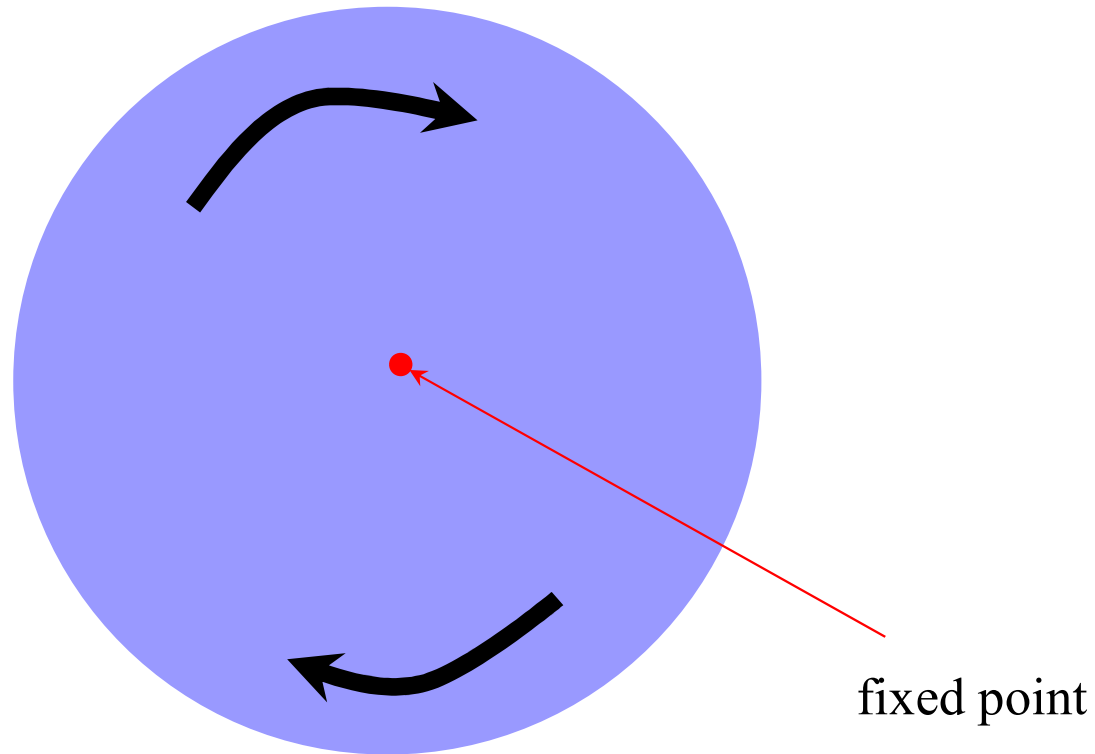


closed and bounded

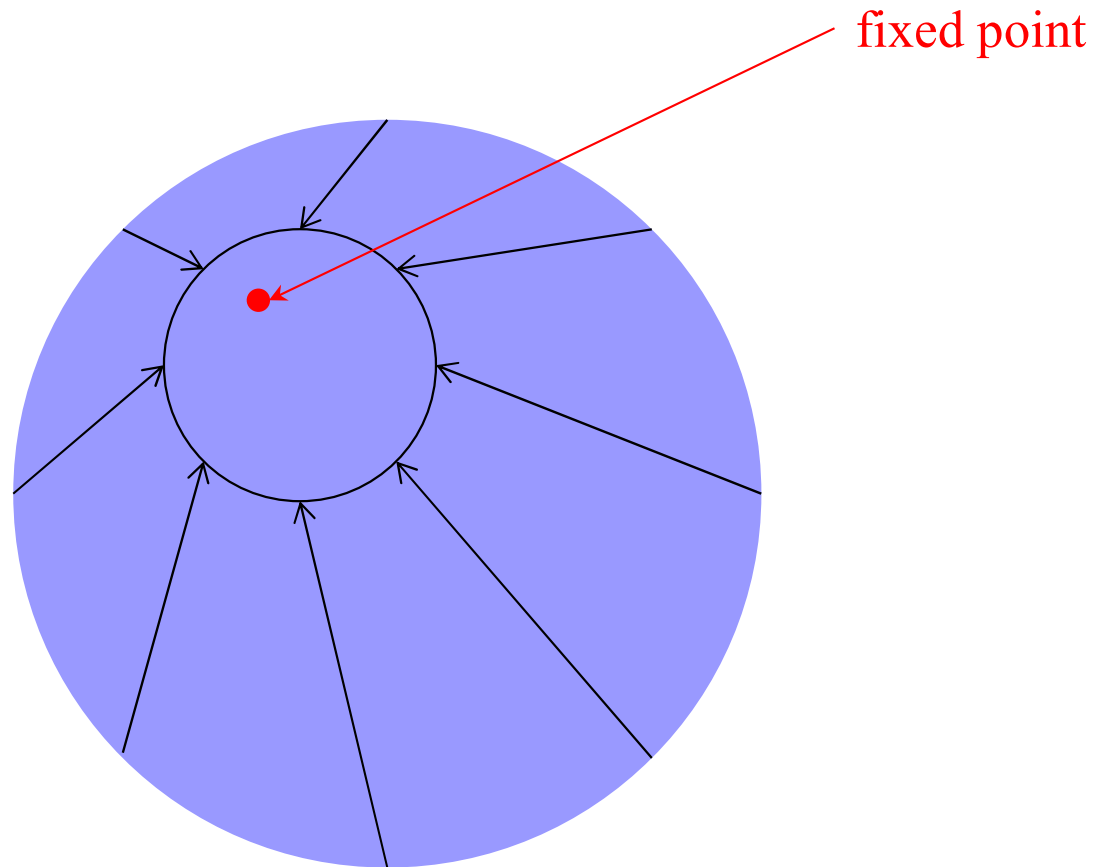
A few examples, when D is the 2-dimensional disk.



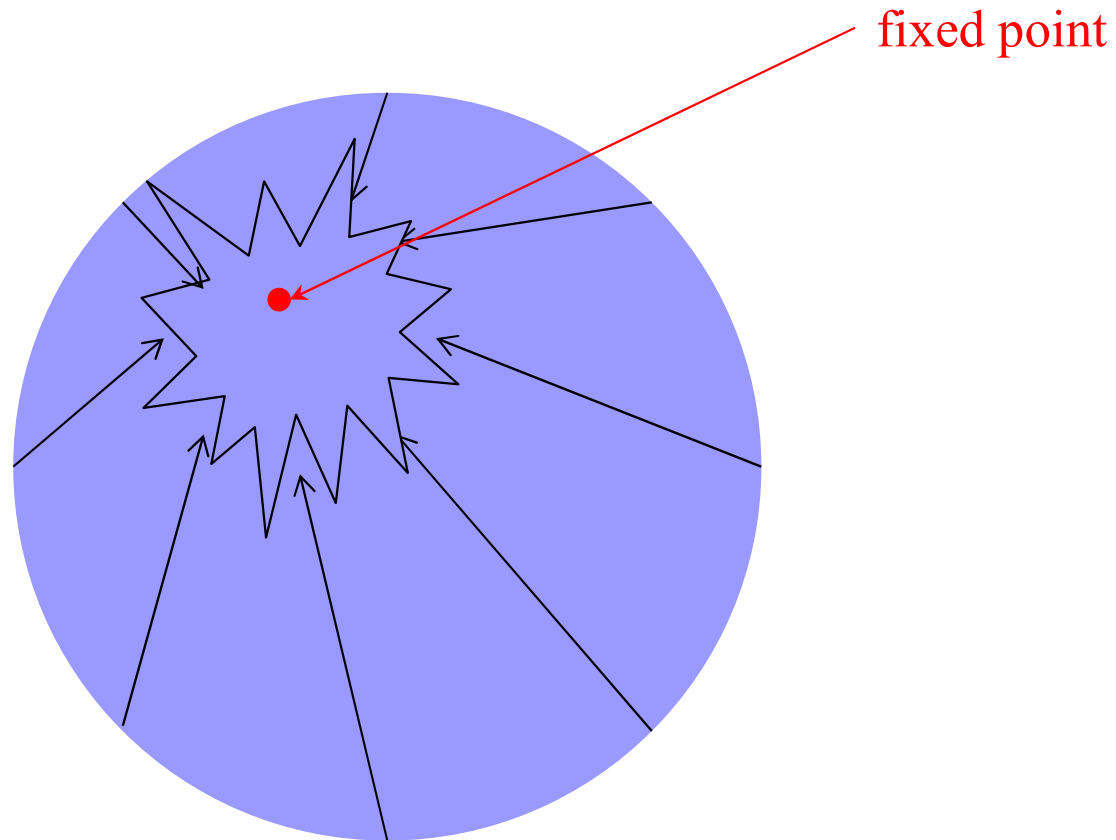
Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem





Brouwer $\Rightarrow$ Nash
(Nash'51)

Nash's Proof:

$$f: \Delta_m \times \Delta_n \rightarrow \Delta_m \times \Delta_n, \quad (x', y') = f(x, y)$$

$$\forall i, \quad \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$$

$$\forall i, \quad x'_i = \frac{x_i + \delta_i}{\sum_{k=1}^n (x_k + \delta_k)}$$

Lemma. If $x' = x$ then x is best for Alice against y
 $\equiv$ If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

$$\forall i, \quad \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$$

$$\forall i, \quad x'_i = \frac{x_i + \delta_i}{\sum_{k=1}^n (x_k + \delta_k)}$$

Lemma. If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

Nash's Proof

$$f: \Delta_m \times \Delta_n \rightarrow \Delta_m \times \Delta_n, \quad (x', y') = f(x, y)$$

$$\forall j, \quad \tau_j = \max\{(x^T B)_j - x^T B y, 0\},$$

$$\forall j, \quad y'_j = \frac{y_j + \tau_j}{\sum_{k=1}^n (y_k + \tau_k)}$$

Lemma. If $y' = y$ then y is best for Bob against x
 $\equiv$ If $x^T B y < x^T B z$ for some $z \in \Delta_n$ then $y' \neq y$.

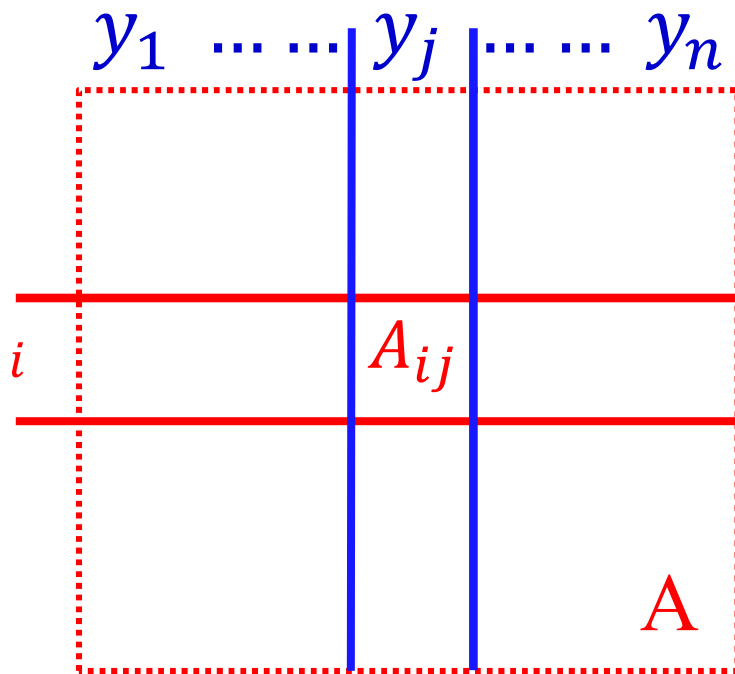


2-Nash Characterization

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- For **Alice**, i^{th} strategy gives

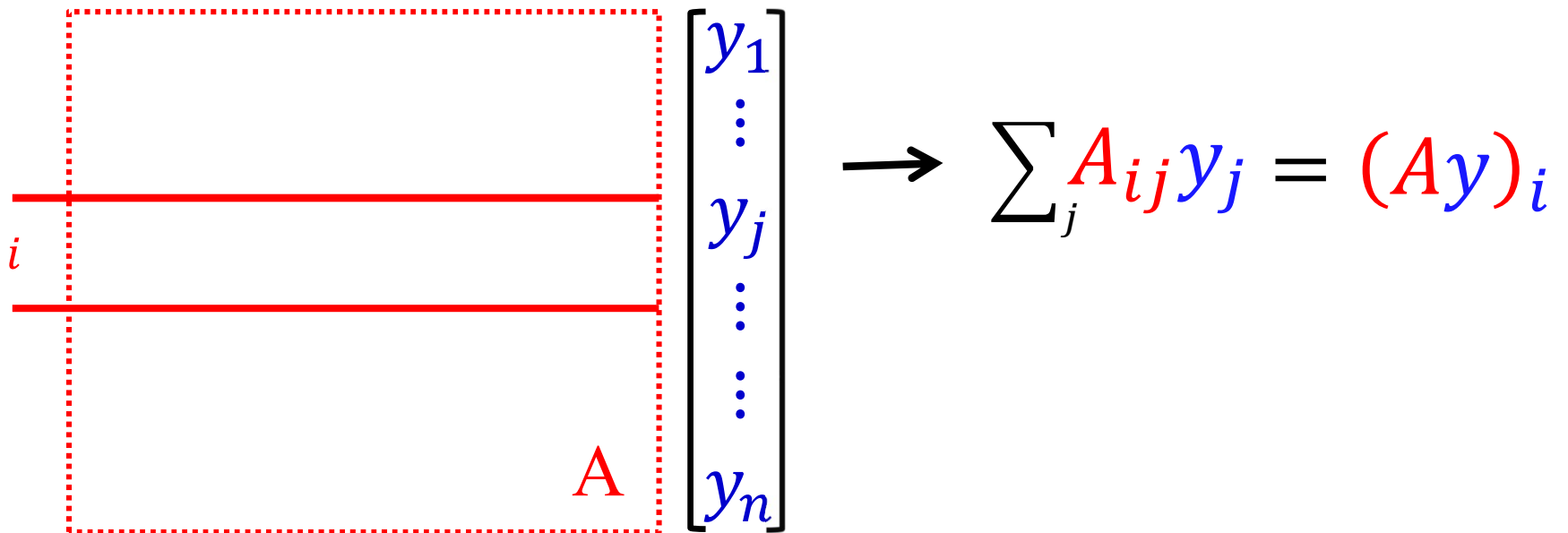


$$\longrightarrow \sum_j A_{ij} y_j$$

2-Nash Characterization



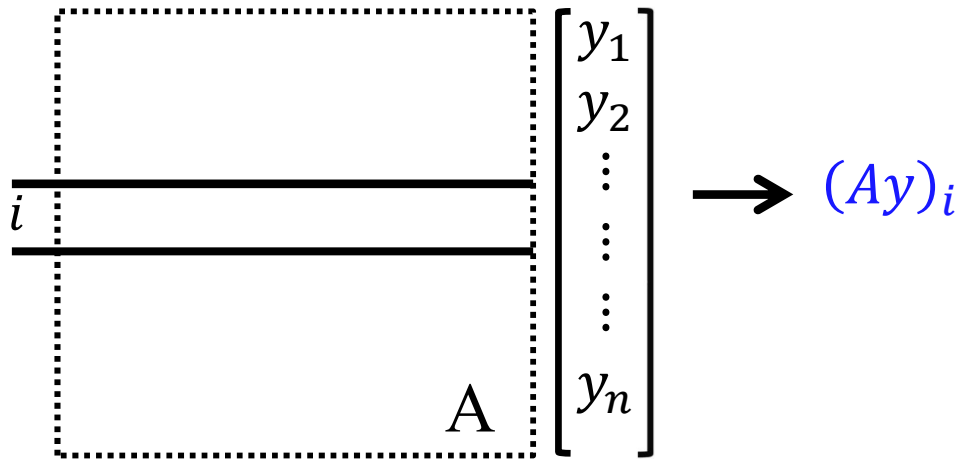
- For **Alice**, i^{th} strategy gives



The diagram illustrates the calculation of the payoff for Alice's i^{th} strategy. A red dotted rectangle encloses a row of the payoff matrix A , with the index i to its left. To the right of this row is a column vector y containing elements $y_1, \dots, y_j, \dots, y_n$. An arrow points from the product of the row and column to the equation $\sum_j A_{ij} y_j = (Ay)_i$.

$$\sum_j A_{ij} y_j = (Ay)_i$$

- i^{th} strategy gives Alice



- Max possible payoff: $\max_i (Ay)_i$

- x achieves max payoff iff

$$x^T A y \geq (Ay)_i, \quad \forall i$$

$$\equiv$$

$$\forall k, \quad x_k > 0 \Rightarrow k \in \operatorname{argmax}_i (Ay)_i$$

Complementarity

- Max possible payoff: $\max_i e_i A y$

- x achieves max payoff iff

$$\forall i, \quad x^T A y \geq (A y)_i$$

$$\equiv$$

$$\forall k, \quad x_k > 0 \Rightarrow (A y)_k = \max_i (A y)_i$$

Complementarity

	H	T
H	1 -1	-2 2
T	-2 2	1 -1

Polyhedra



max-payoff $\leq \pi_A$

$$P \quad \boxed{\begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}} \quad \begin{array}{l} y_1, \dots, y_n \geq 0 \\ \sum_{j=1}^n y_j = 1 \end{array}$$



max-payoff $\leq \pi_B$

$$Q \quad \boxed{\begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}} \quad \begin{array}{l} x_i \geq 0 : i=1 \dots m \\ \sum_{i=1}^m x_i = 1 \end{array}$$



P

$$\forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n$$



Q

$$\forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m$$

$$(y, \pi_A) \in P, \quad (x, \pi_B) \in Q$$

Sum of payoffs

At least the sum of
max payoffs

$$\overbrace{x^T (A + B) y}^{\text{Sum of payoffs}} - \overbrace{(\pi_A + \pi_B)}^{\text{At least the sum of max payoffs}} \leq 0$$

$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

$$(y, \pi_A) \in P, \quad (x, \pi_B) \in Q$$

Sum of payoffs

At least the sum of
max payoffs

$$x^T (A + B) y - (\pi_A + \pi_B) = 0$$

easy $\uparrow$



$\downarrow$ exe

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T (A + B)y - (\pi_A + \pi_B) \leq 0$.

(ii) $x^T (A + B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T (A + B)y - (\pi_A + \pi_B) \leq 0$.

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$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

$$(y, \pi_A) \in P, \quad (x, \pi_B) \in Q$$

Sum of
2-Nash payoffs

At least the sum of
max payoffs

$$\max: x^T (A + B) y - (\pi_A + \pi_B) \quad = 0$$

$$\text{s.t. } (y, \pi_A) \in P, (x, \pi_B) \in Q \quad \text{Complementarity}$$

1. (x, y) is a NE
2. π_A and π_B are the max payoffs


$$A_{ij} = -B_{ij} \Rightarrow A_{ij} + B_{ij} = 0 \quad \forall (i,j) \\ \Rightarrow A+B = [\bar{0}]$$

Zero-sum Games

Von Neuman's maxmin theorem (1928) = LP-duality

$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

$$(y, \pi_A) \in P, \quad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash $\rightarrow$ linear programming

$$\max: x^T \overset{[0]}{\cancel{(A+B)}} y - (\pi_A + \pi_B)$$

$$\text{s.t. } (y, \pi_A) \in P, \quad (x, \pi_B) \in Q$$


$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

$$(y, \pi_A) \in P, \quad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then
2-Nash $\rightarrow$ linear programming

$$\begin{array}{ll} \text{max:} & -(\pi_A + \pi_B) \\ \text{s.t.} & (y, \pi_A) \in P, \quad (x, \pi_B) \in Q \end{array}$$

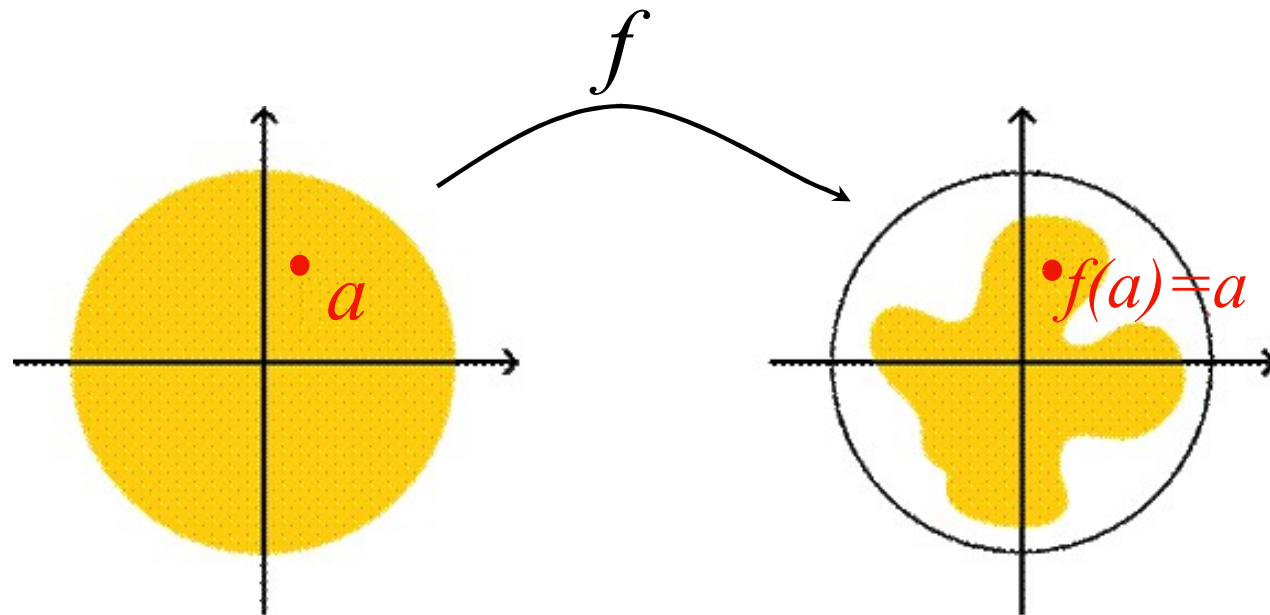
Theorem. [von Neumann'28] (max-min = min-max) Game $(A, -A)$.

Wrt A , Alice is a maximizer and Bob minimizer. Then,

$$\max_x \min_y x^T A y = \min_y \max_x x^T A y \quad \& \text{ the max-min is NE.}$$

Computation in general?

NE existence via fixed-point theorem.



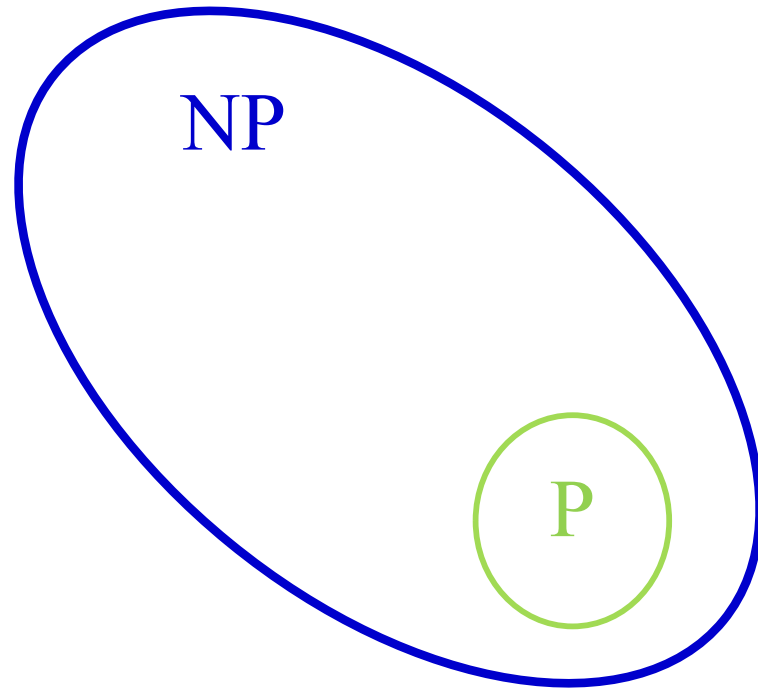
Computation? (in Econ)

- Special cases: Dantzig'51, Lemke-Howson'64, Elzen-Talman'88, Govindan-Wilson'03, ...
- Scarf'67: Approximate fixed-point.
 - Numerical instability
 - Not efficient!
- ...

Computation? (in CS)

Not easy!

$\exists$ solution?



What if solution always exists, like Nash Eq.?

Computation? (in CS)

Megiddo and Papadimitriou'91 :

Nash is NP-hard $\Rightarrow$ NP=Co-NP

NP-hardness is ruled out!

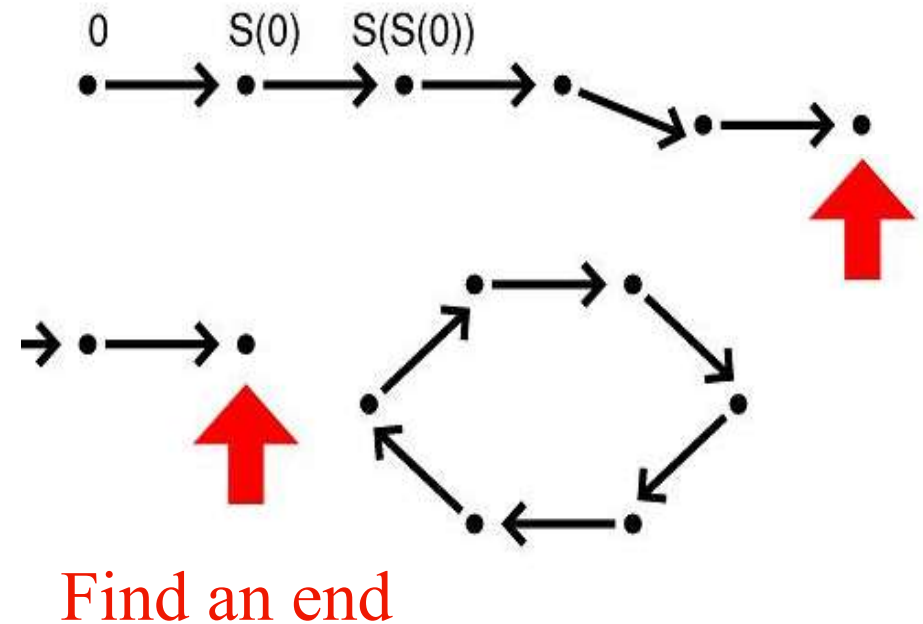
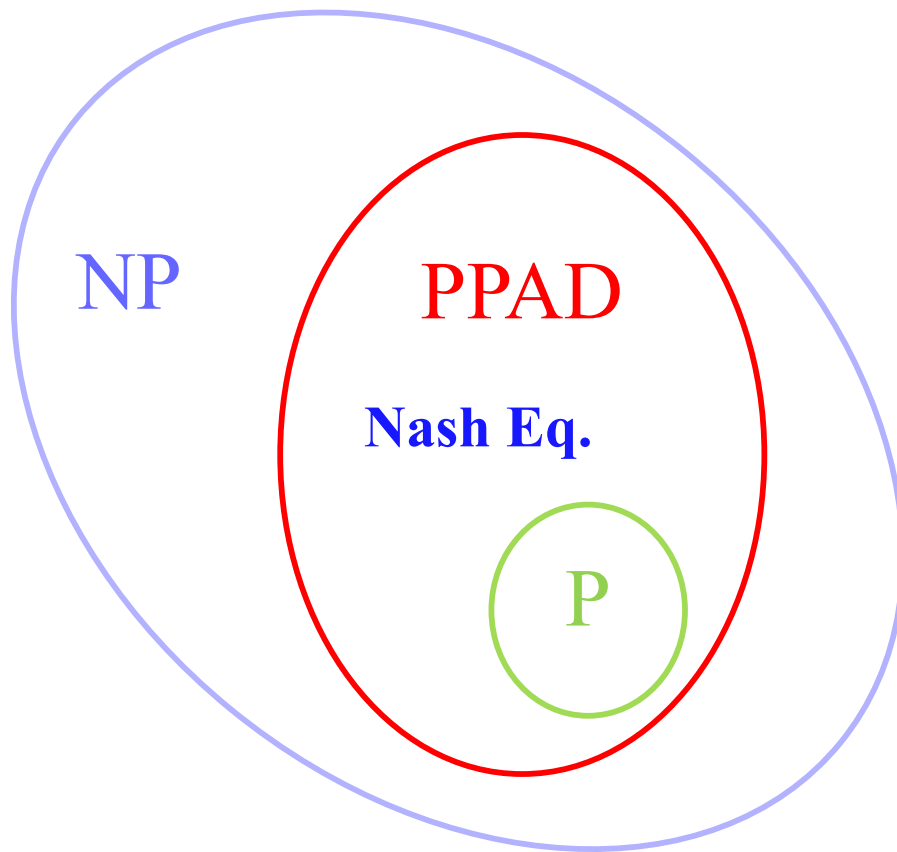
Complexity Classes

Papadimitriou'94

PPAD Polynomial Parity Argument for Directed graph

2-Nash is PPAD-complete!

[DGP'06, CDT'06]



Brute-force Algorithm?

$$P \quad \boxed{\begin{array}{l} \forall i, (A\mathbf{y})_i \leq \pi_A \\ \mathbf{y} \in \Delta_n \end{array}}$$

$$Q \quad \boxed{\begin{array}{l} \forall j, (\mathbf{x}^T B)_j \leq \pi_B \\ \mathbf{x} \in \Delta_m \end{array}}$$

Let (x, y) be a NE. Suppose we know $\text{supp}(x)$ and $\text{supp}(y)$.

Now can we find a NE?



Can we do better than “brute-force”?

Not so far. And may be never!

It is one of the hardest problems in PPAD.



What about special cases/approximation?

- $\text{Rank}(A)$ or $\text{rank}(B)$ is constant
- $O(1)$ -approximate NE: quasi-polynomial time algorithm
- Constant rank games: $\text{rank}(A+B)$ is a constant
 - FPTAS

$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

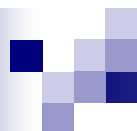
$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then
 2-Nash $\rightarrow$ linear programming

$$\begin{array}{ll} \text{max:} & -(\pi_A + \pi_B) \\ \text{s.t.} & (y, \pi_A, x, \pi_B) \in P \times Q \end{array}$$

Rank of a game: rank(A+B)

Zero-sum $\equiv$ Rank-0 games


$$P \quad \begin{array}{l} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$$

$$Q \quad \begin{array}{l} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then
2-Nash $\rightarrow$ linear programming

Rank of a game: $\text{rank}(A+B)$

Poly-time approximation for constant rank games
[KT'03].

Poly-time exact for rank-1 games [AGMS'11].

Exact for rank > 2 is PPAD-hard [M'13].