Games and Nash Equilibrium

CS 580

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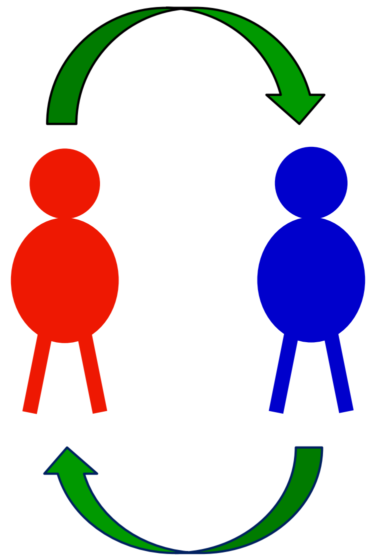
Agenda

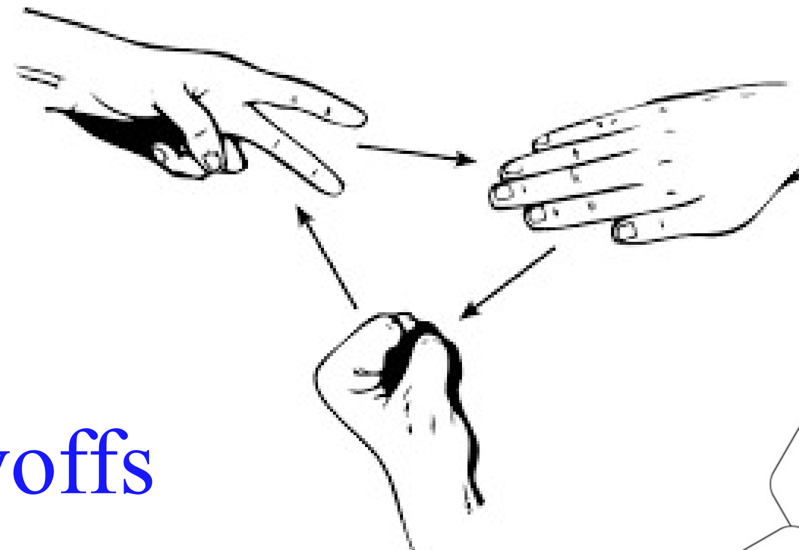
- Two-player Games
- Nash Equilibrium (NE)
  - NE existence
  - NE characterization
- Zero-sum games
  - Minmax Theorem
  - LP-duality

Games

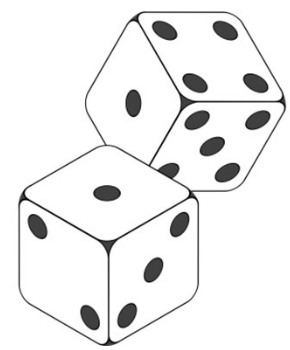


Players

Payoffs

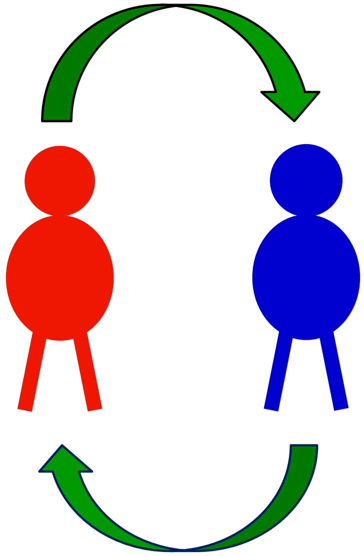


Strategies

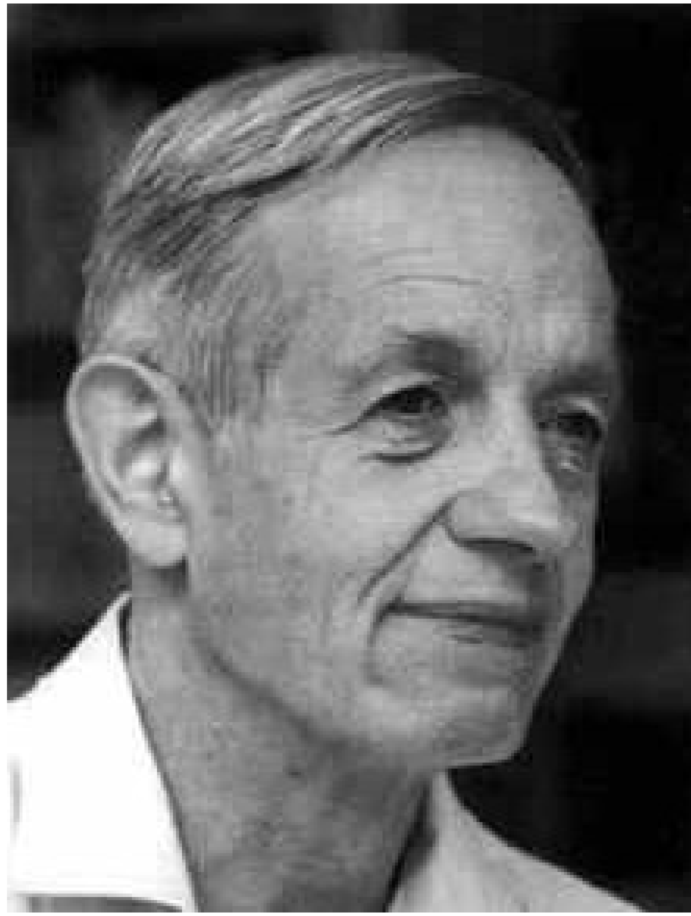


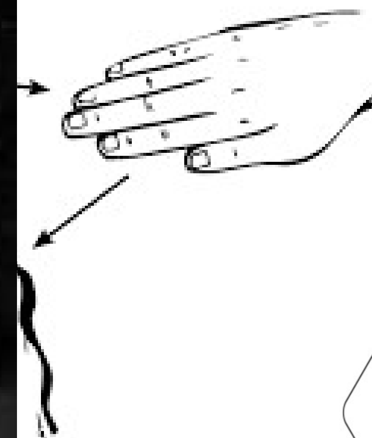
Randomize!

Games

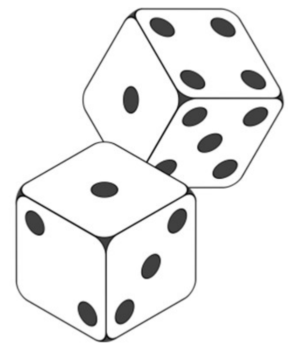


Players





tegies



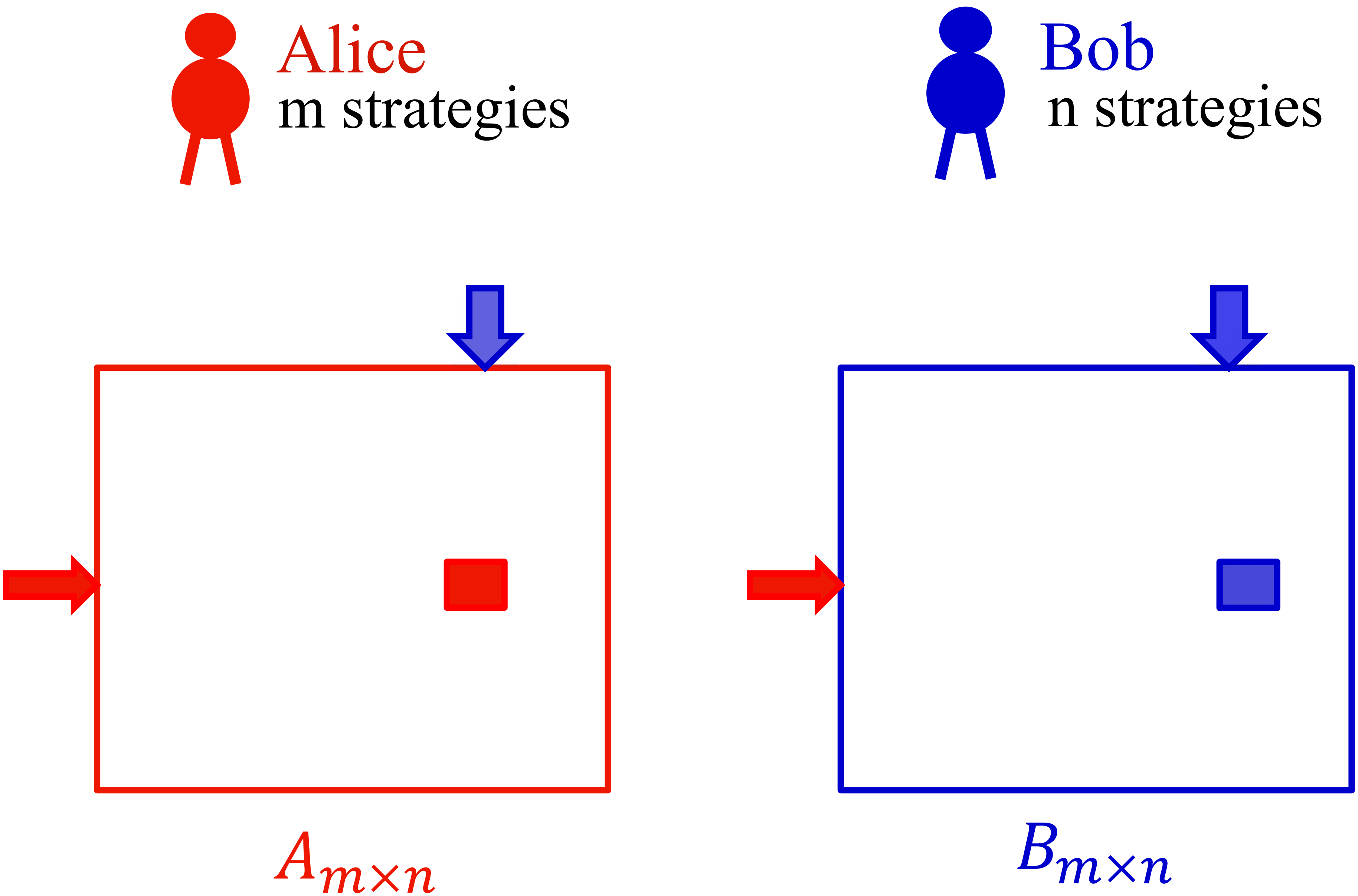
Randomize!

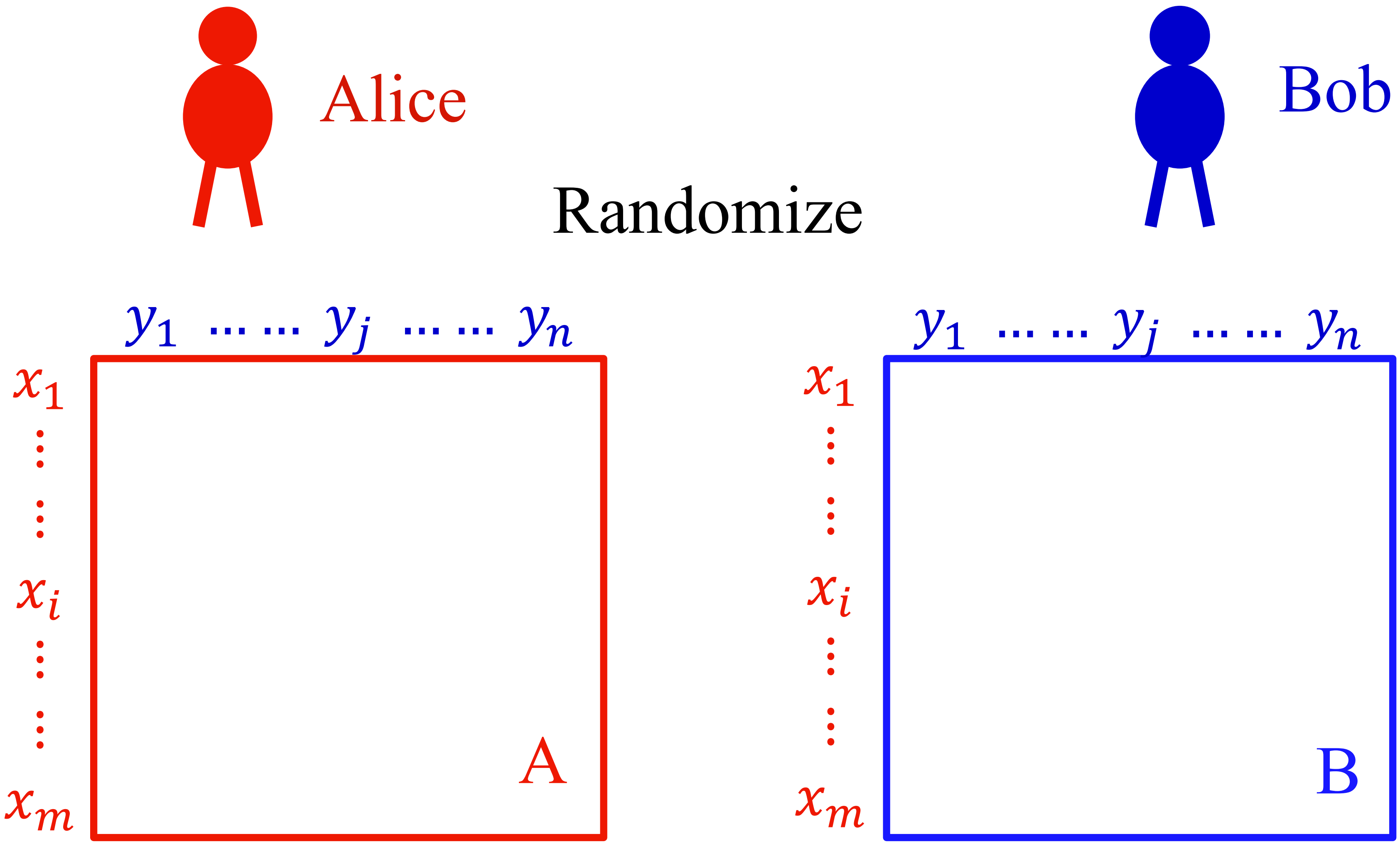
Nash (1950):

There exists a (stable) state where no player gains by unilateral deviation.

Nash equilibrium (NE)

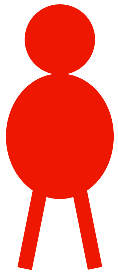
Our focus: Two-player games





Alice
Bob
Randomize
y_1 …… y_j …… y_n
x_1
x_i
x_m
A
y_1 …… y_j …… y_n
x_1
x_i
x_m
B

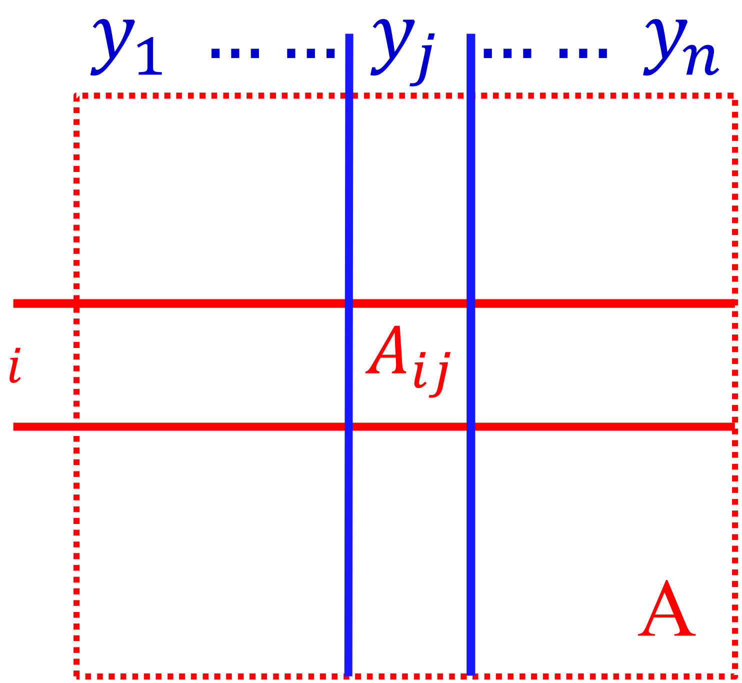
2-Nash Characterization



Alice

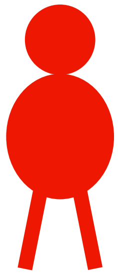
Bob

- For Alice, i^{th} strategy gives



$$\longrightarrow \sum_j A_{ij} y_j$$

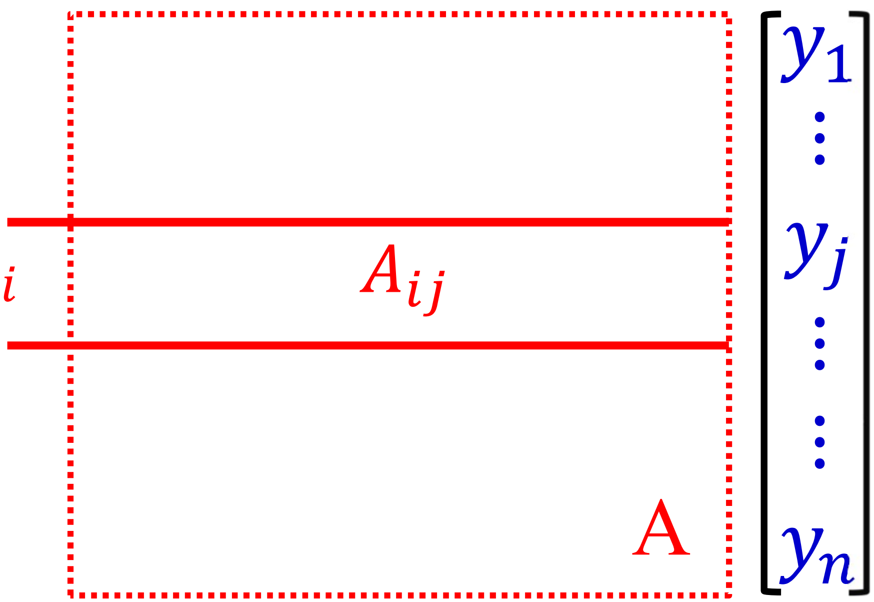
2-Nash Characterization



Alice

Bob

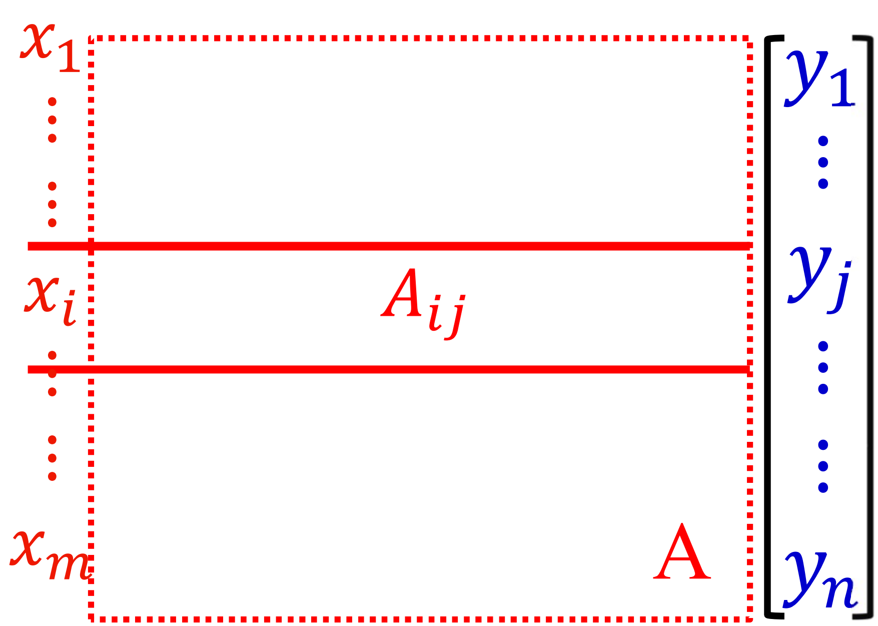
- For Alice, i^{th} strategy gives



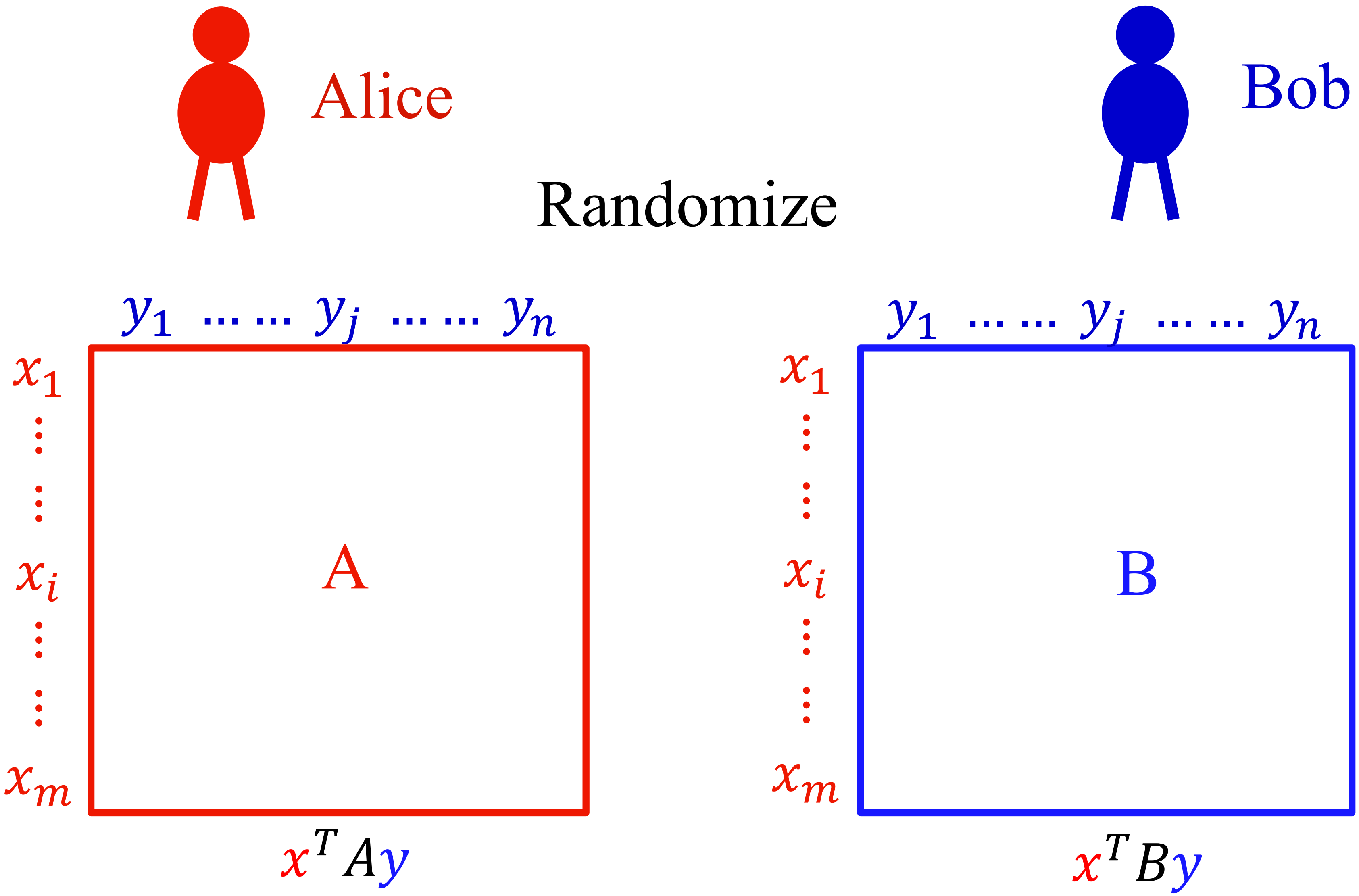
$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

2-Nash Characterization

- Alice's expected payoff is



$$\sum_i x_i (Ay)_i = x^T A y$$



NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \qquad \forall z \in \Delta_m$$

$$x^T B y \geq x^T B z, \qquad \forall z \in \Delta_n$$

Example: Matching Pennies

	H	T
H	1 -1	-2 2
T	-2 2	1 -1

Nash's Existence Theorem (1950)

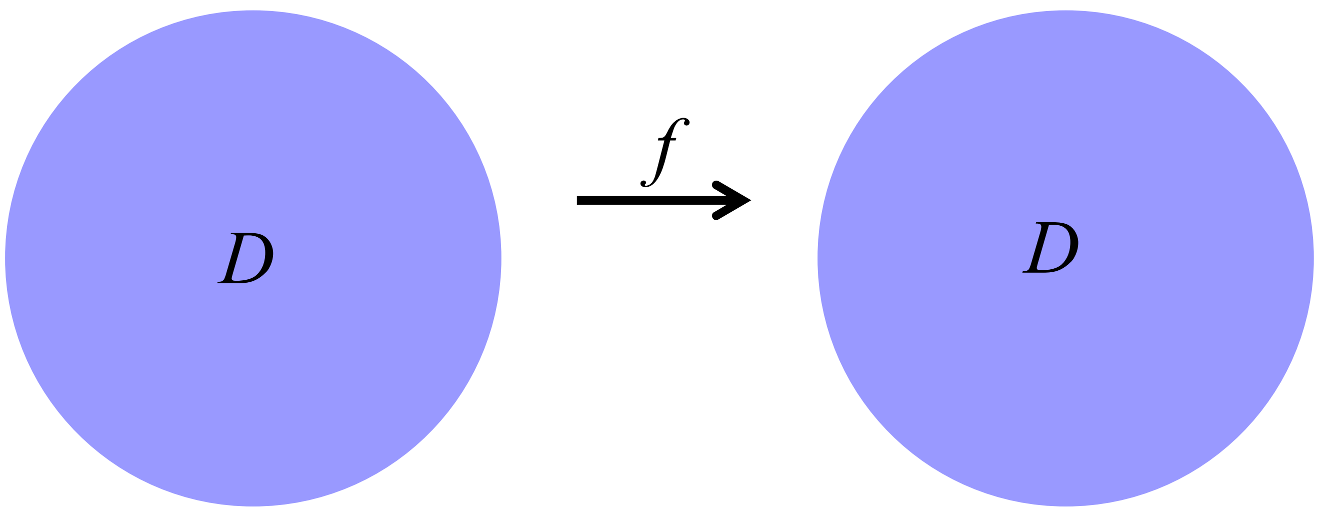
Brouwer's Fixed Point Theorem

[Brouwer 1910]: Let $f : D \to D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

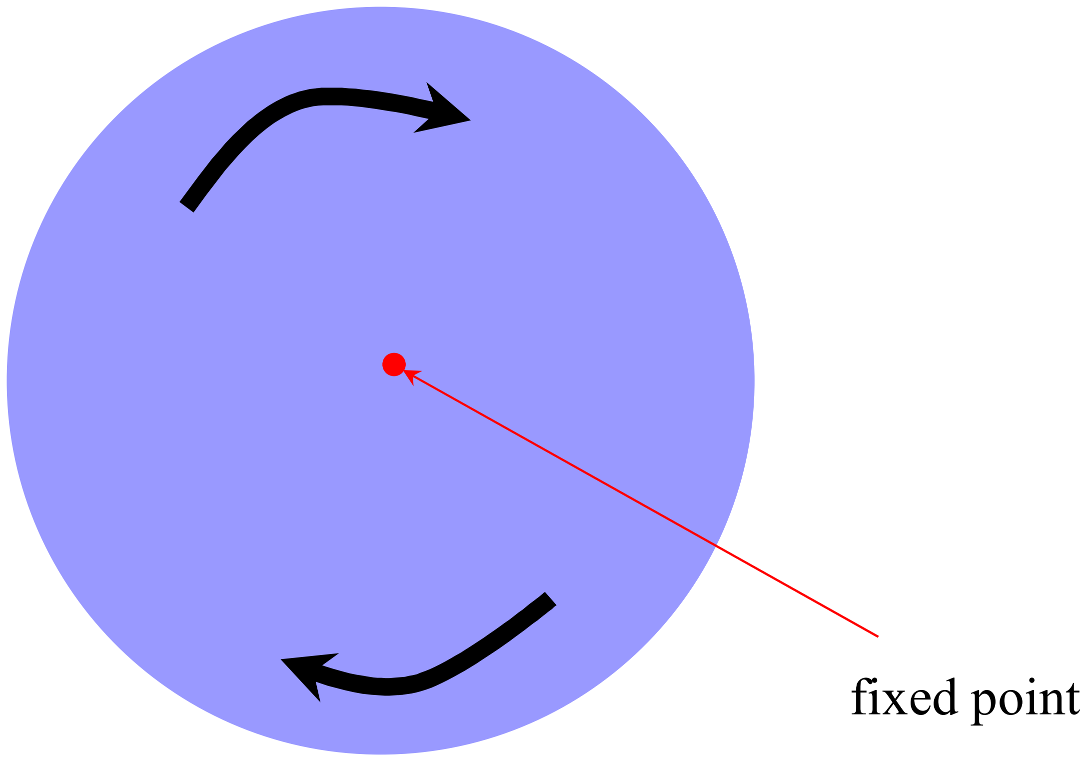
Then there exists an $x \in D$ s.t. $x = f(x)$.

closed and bounded

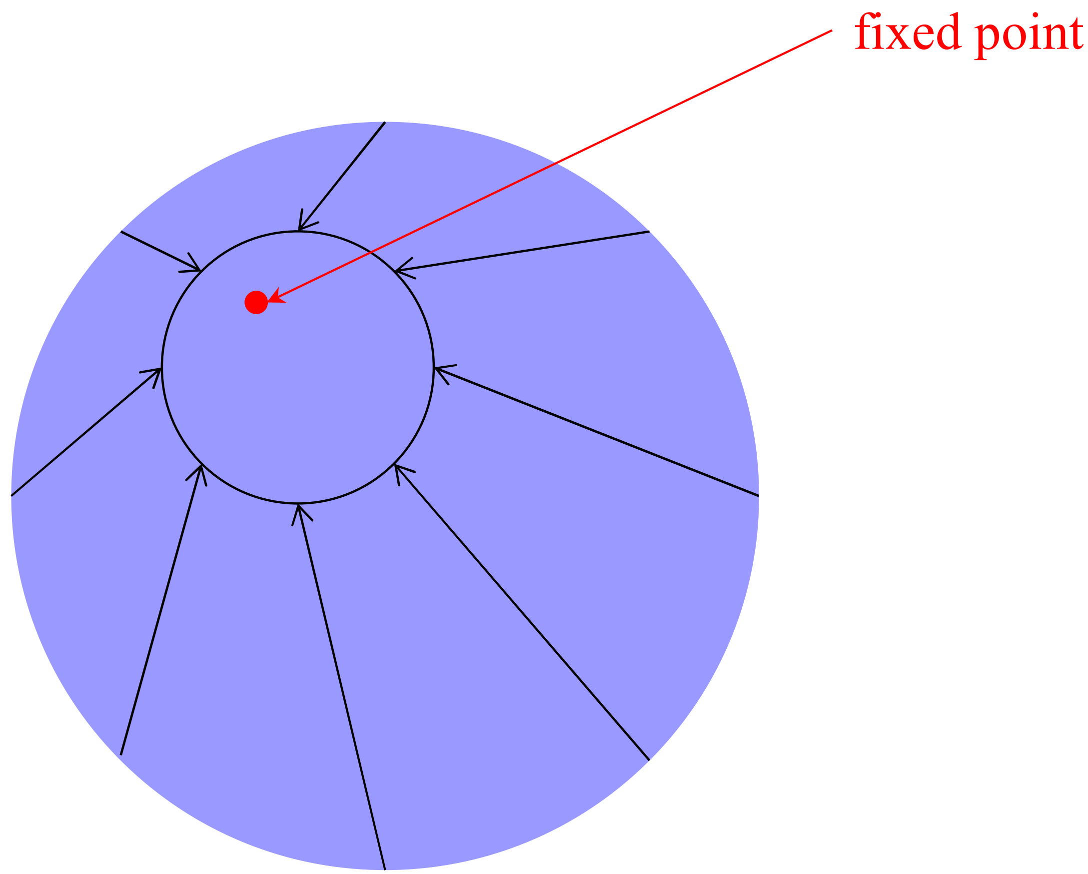
A few examples, when D is the 2-dimensional disk.



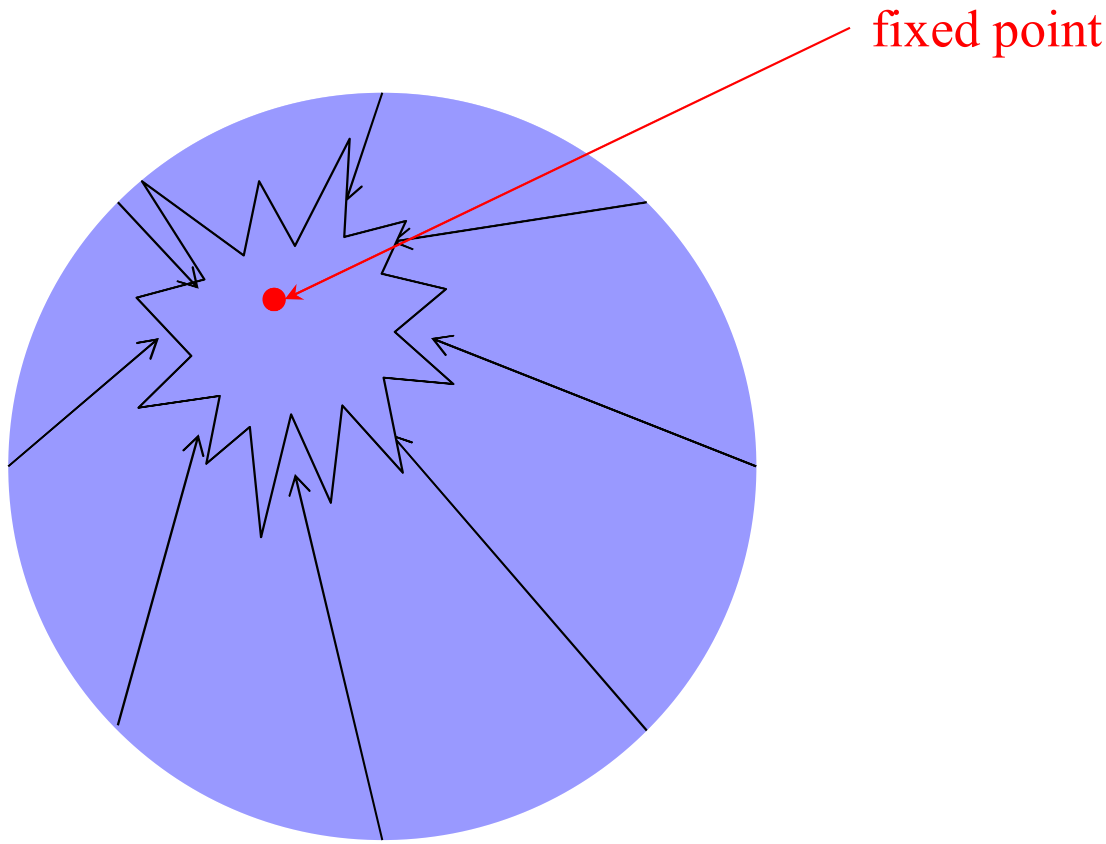
Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem



$$Brouwer \Rightarrow Nash$$

(Nash'51)

Nash's Proof:

$f: \Delta_m \times \Delta_n \to \Delta_m \times \Delta_\mathrm{n}$, $\quad (x', y') = f(x, y)$

$\forall i, \quad \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$

$$\forall i, \quad x_i' = \frac{x_i + \delta_i}{\sum_{k=1}^{n}(x_k + \delta_k)}$$

Lemma. If $x' = x$ then x is best for Alice against y

$\equiv$ If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

$\forall i, \qquad \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$

$$\forall i, \qquad x_i' = \frac{x_i + \delta_i}{\sum_{k=1}^{n}(x_k + \delta_k)}$$

Lemma. If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

Nash's Proof

$f: \Delta_m \times \Delta_n \to \Delta_m \times \Delta_\mathrm{n}$, $(x', y') = f(x, y)$

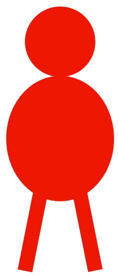
$\forall j,$ $\tau_j = \max\{(x^T B)_j - x^T B y, 0\},$

$$\forall j, \qquad y'_j = \frac{y_j + \tau_j}{\sum_{k=1}^{n}(y_k + \tau_k)}$$

Lemma. If $y' = y$ then y is best for Bob against x
$\equiv$ If $x^T B y < x^T B z$ for some $z \in \Delta_n$ then $y' \neq y$.

2-Nash Characterization

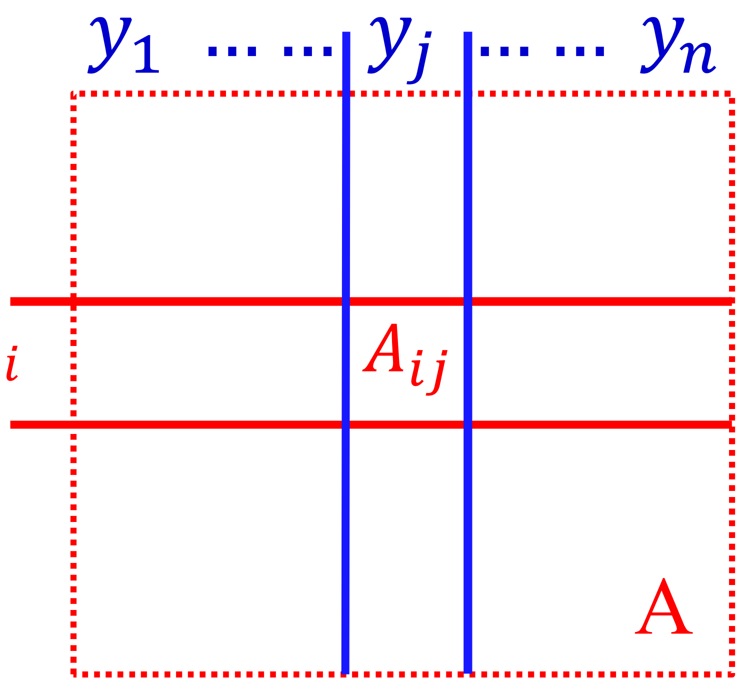
2-Nash Characterization



Alice

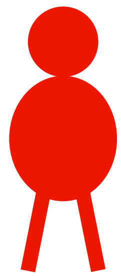
Bob

- For Alice, i^{th} strategy gives



$$\longrightarrow \sum_j A_{ij} y_j$$

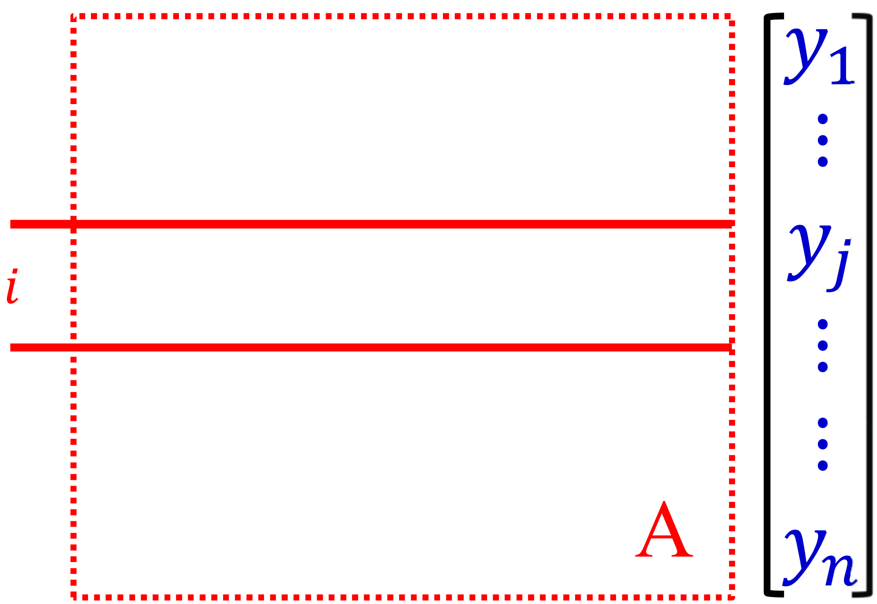
2-Nash Characterization



Alice

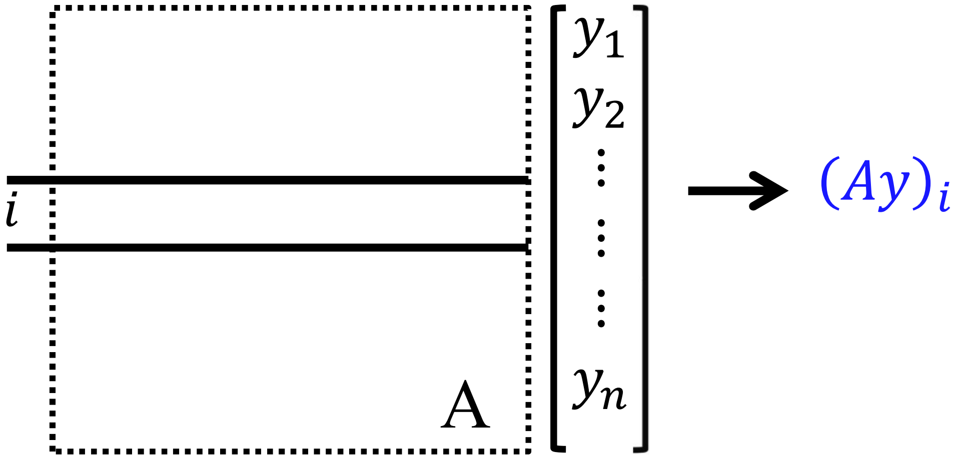
Bob

- For Alice, i^{th} strategy gives



$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

- i^{th} strategy gives Alice



- Max possible payoff: $\max_i (Ay)_i$

- x achieves max payoff iff

$$x^T Ay \geq (Ay)_i, \qquad \forall i$$

$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow k \in \operatorname{argmax}_i (Ay)_i$$

Complementarity

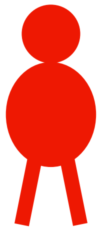
- Max possible payoff: $\max_i e_i A y$
- x achieves max payoff iff

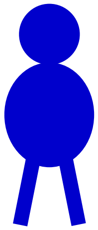
$$\forall i, \qquad x^T A y \geq (Ay)_i$$
$$\equiv$$
$$\forall k, \qquad x_k > 0 \Rightarrow (Ay)_k = \max_i (Ay)_i$$

Complementarity

	H	T
H	1 -1	-2 2
T	-2 2	1 -1

Polyhedra





max-payoff $\leq \pi_A$

max-payoff $\leq \pi_B$

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}} \quad \begin{array}{l} y_1, \dots, y_n \geq 0 \\ \sum_j y_j = 1 \end{array}$$

$$Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}} \quad \begin{array}{l} x_i \geq 0 : i = 1 \dots m \\ \sum_{i=1}^{m} x_i = 1 \end{array}$$

P

$$\forall i, (Ay)_i \le \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, (x^T B)_j \le \pi_B$$
$$x \in \Delta_m$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

$$\overbrace{x^T(A+B)y}^{\text{Sum of payoffs}} - \overbrace{(\pi_A + \pi_B)}^{\text{At least the sum of max payoffs}} \le 0$$

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}}$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

$$\overbrace{x^T(A+B)y}^{\text{Sum of payoffs}} - \overbrace{(\pi_A + \pi_B)}^{\text{At least the sum of max payoffs}} \boxed{= \mathbf{0}}$$

easy $\Uparrow$ $\Updownarrow$ $\Downarrow$ exe Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}}$$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T(A+B)y - (\pi_A + \pi_B) \leq 0$.

(ii) $x^T(A+B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

P $\boxed{\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}}$

Q $\boxed{\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}}$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

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P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$ Q $\boxed{\begin{array}{c}\forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

$(y, \pi_A) \in P, \quad (x, \pi_B) \in Q$

2-Nash

Sum of payoffs

At least the sum of max payoffs

$$\text{max: } \underbrace{x^T(A+B)y}_{\text{Sum of payoffs}} - \underbrace{(\pi_A + \pi_B)}_{\text{At least the sum of max payoffs}} \quad \boxed{\mathbf{= 0}}$$

s.t. $(y, \pi_A) \in P, (x, \pi_B) \in Q$

$\Updownarrow$

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

$A_{ij} = -B_{ij} \Rightarrow A_{ij} + B_{ij} = 0 \quad \forall (i,j)$

$\Rightarrow A + B = [\bar{0}]$

Zero-sum Games

Von Neuman's maxmin theorem (1928) = LP-duality

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}}$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\text{max: } x^T \overset{[0]}{\cancel{(A + B)}} y - (\pi_A + \pi_B)$$

$$\text{s.t. } (y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}}$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

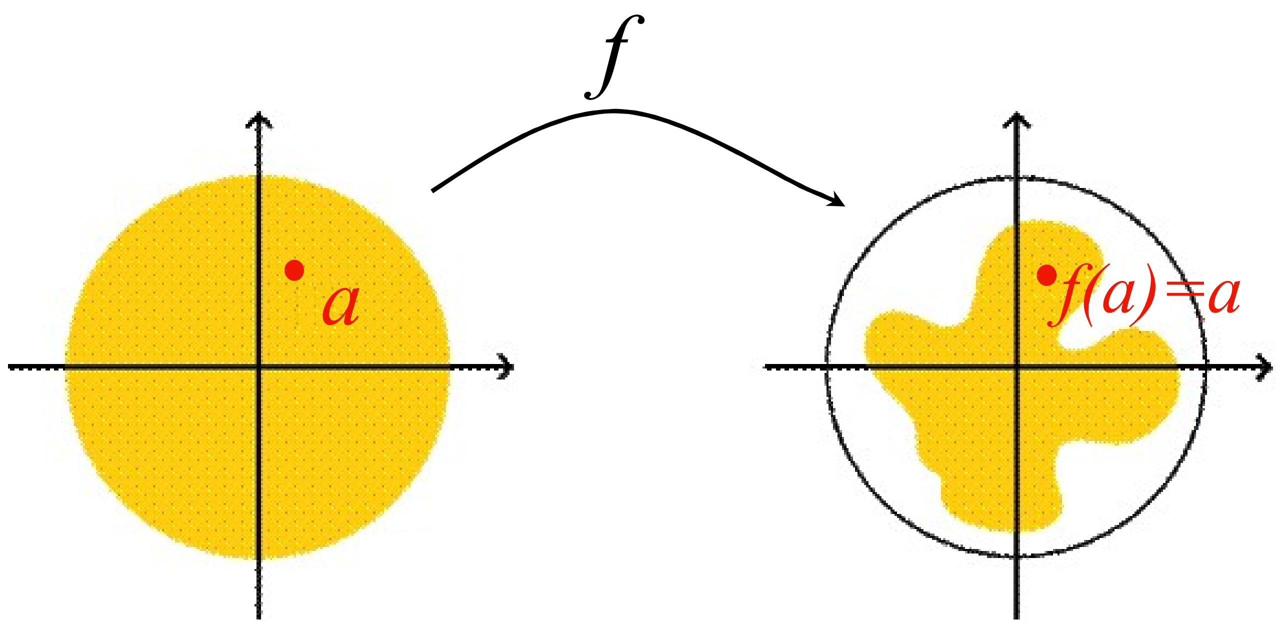
$$\begin{aligned} &\text{max: } -(\pi_A + \pi_B) \\ &\text{s.t. } (y, \pi_A) \in P, \qquad (x, \pi_B) \in Q \end{aligned}$$

Theorem. [von Neumann'28] (max-min = min-max) Game $(A, -A)$. Wrt A, Alice is a maximizer and Bob minimizer. Then,

$$\max_x \min_y x^T A y = \min_y \max_x x^T A y$$ & the max-min is NE.

Computation in general?

NE existence via fixed-point theorem.



Computation? (in Econ)

- Special cases: Dantzig'51, Lemke-Howson'64, Elzen-Talman'88, Govindan-Wilson'03, …

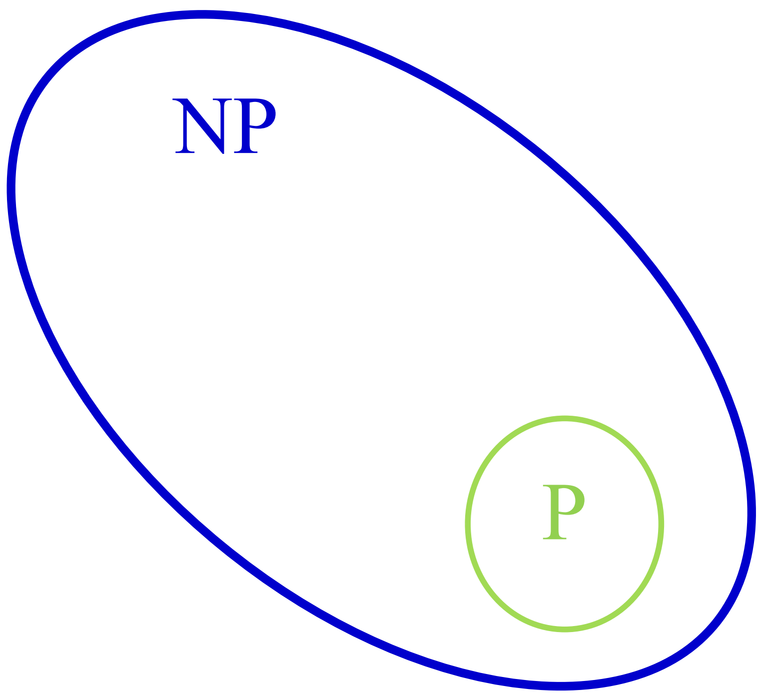
- Scarf'67: Approximate fixed-point.
  - Numerical instability
  - Not efficient!

- …

Computation? (in CS)

Not easy!

∃ solution?



What if solution always exists, like Nash Eq.?

Computation? (in CS)

Megiddo and Papadimitriou'91 :

Nash is NP-hard $\Rightarrow$ NP=Co-NP

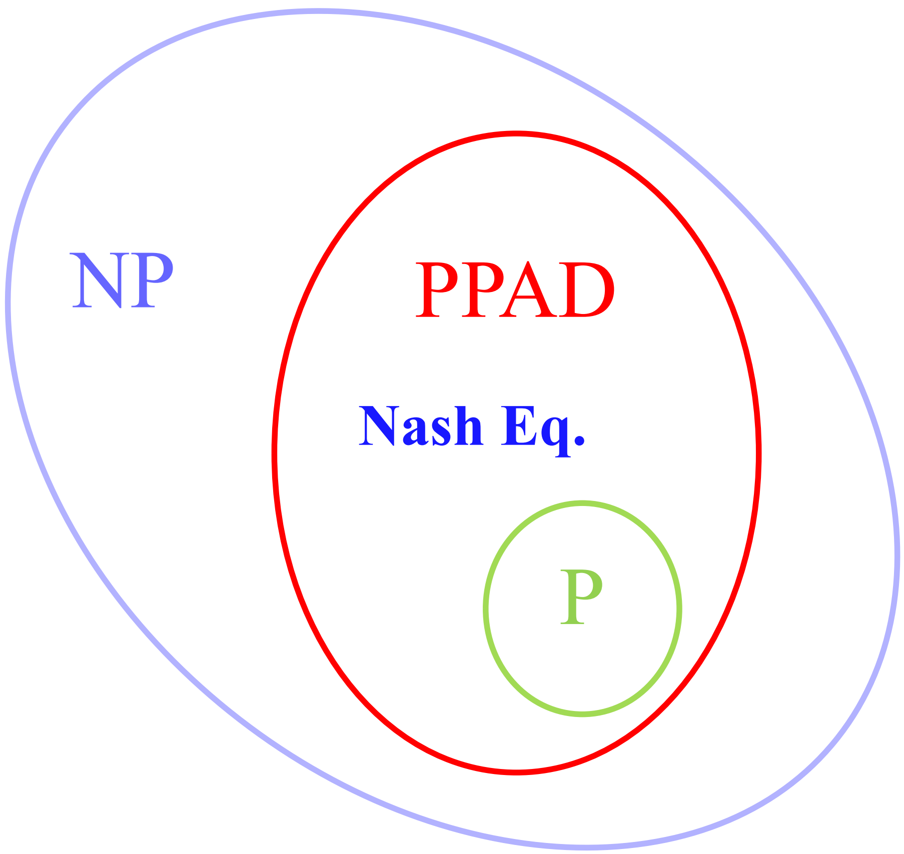
NP-hardness is ruled out!

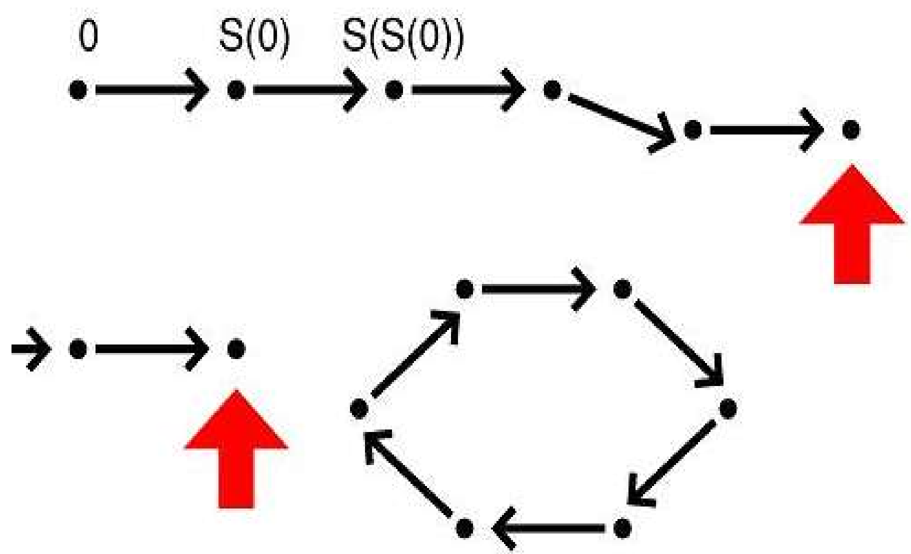
Complexity Classes

2-Nash is PPAD-complete!
[DGP'06, CDT'06]

Papadimitriou'94

PPAD   Polynomial Parity Argument for Directed graph





Find an end

Brute-force Algorithm?

P $\boxed{\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}}$

Q $\boxed{\begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}}$

Let (x, y) be a NE. Suppose we know supp(x) and supp(y).

Now can we find a NE?

Can we do better than “brute-force”?

Not so far. And may be never!

It is one of the hardest problems in PPAD.

What about special cases/approximation?

- Rank(A) or rank(B) is constant
- O(1)-approximate NE: quasi-polynomial time algorithm
- Constant rank games: rank(A+B) is a constant
  - FPTAS

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}}$$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\max: -(\pi_A + \pi_B)$$

$$\text{s.t. } (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank of a game: rank(A+B)

Zero-sum ≡ Rank-0 games

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c} \forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m \end{array}}$$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

Rank of a game: rank(A+B)

Poly-time approximation for constant rank games [KT'03].

Poly-time exact for rank-1 games [AGMS'11].

Exact for rank > 2 is PPAD-hard [M'13].