Fair Division of Indivisible Items (Part II)

CS 580

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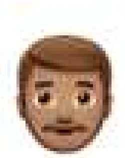
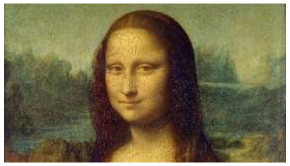
Most slides are curtesy Prof. J. Garg

- N: set of n agents, 1 ,…, n,
- M: set of m indivisible items (like cell phone, painting, etc.)





- Agent i has a valuation function $v_i : 2^m \to \mathbb{R}$ over subsets of items
  - Monotone: the more the happier

Last Lecture

- EF: Envy-free, Prop: Proportional
  - Do not exist
- EF1: Envy-free up to one item.
  - Round Robin for additive valuations
  - Envy-cycle elimination for general monotone
- Prop1: Proportional up to one item
  - EF1 implies Prop1 under additive valuations
  - CE + Rounding algorithm for general valuations.
- Open:
  - EF1+PO: Polytime computation for additive, existence beyond additive.
  - Chores: EF1+PO
  - …

EFX: Envy-free up to *any* item

Envy-Freeness up to One Item (EF1)

- An allocation $(A_1, \dots, A_n)$ is EF1 if for every agent i

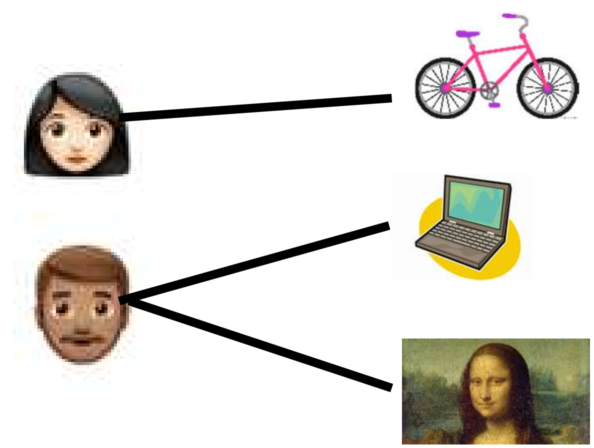
$\forall g \leftarrow$ EFX

$$\forall k \in N, \qquad v_i(A_i) \geq v_i(A_k \setminus g), \qquad \cancel{\exists g} \in A_k.$$

That is, agent i may envy agent k, but the envy can be eliminated if we remove a single item from $k's$ bundle

[10, 10, 20]

[1, 20, 10]



Envy-Freeness up to Any Item (EFX) [CKMPS14]

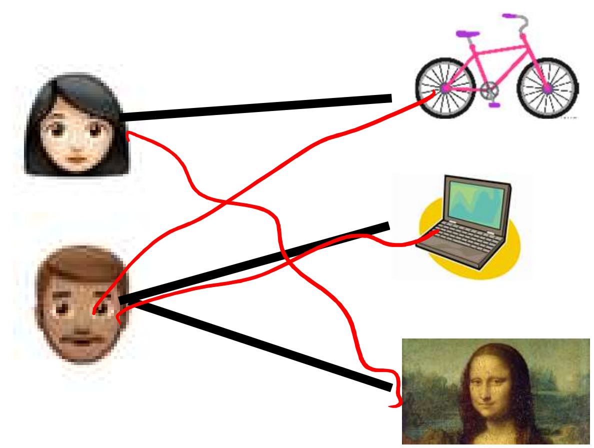
- An allocation $(A_1, \ldots, A_n)$ is EFX if for every agent i

$$\forall k \in N, \qquad v_i(A_i) \geq v_i(A_k \setminus g), \qquad \forall g \in A_k.$$

That is, agent i may envy agent k, but the envy can be eliminated if we remove any single item from $k's$ bundle

EF1 [10, 10, 11]

EFX ? [1, 10, 10]



EFX: Existence

- General Valuations [PR18]
  - $n = 2$
  - Identical Agents

- $n = 3$: Does not exist (even when $m \geq (n+5)$)! [AMMSW'26]

- Additive Valuations
  - $n = 3$ [CGM20]

Additive ($n > 3$), OXS ($n > 2$)

"Fair division's one of the biggest problem" [P20]

Summary

Covered

- EF1 (existence/polynomial-time algorithm)
- EF1 + PO (partially)
- EFX (partially)
- Prop1

Not Covered

- EFX for 3 (additive) agents
- Partial EFX allocations
  - Little Charity [CKMS20, CGMMM21]
  - High Nash welfare [CGH19]
- Chores
  - EF1 (existence/ polynomial-time algorithm) EXERCISE

Major Open Questions (additive valuations)

- EF1+PO: Polynomial-time algorithm
- EF1+PO: Existence for chores **(Resolved recently!)**
- EFX : $n \geq 3$ Existence / Non-existence

Proportionality

- A set N of n agents, a set M of m indivisible goods

- Proportionality: Allocation $A = (A_1, \dots, A_n)$ is proportional if each agent gets at least $1/n$ share of all items:

$$v_i(A_i) \geq \frac{v_i(M)}{n}, \qquad \forall i \in N$$

Cut-and-choose?

Maximin Share (MMS) [B11]

Cut-and-choose.

- Suppose we allow agent i to propose a partition of items into n bundles with the condition that i will choose at the end.
- Clearly, i partitions items in a way that maximizes the value of her least preferred bundle.
- μ_i := Maximum value of $i's$ least preferred bundle

Maximin Share (MMS) [B11]

Cut-and-choose.

- Suppose we allow agent i to propose a partition of items into n bundles with the condition that i will choose at the end
- Clearly, i partitions items in a way that maximizes the value of her least preferred bundle
- $\mu_i :=$ Maximum value of $i's$ least preferred bundle

- $\Pi \coloneqq$ Set of all partitions of items into n bundles
- $\mu_i \coloneqq \max_{(A_1,\ldots,A_n)\in\Pi} \min_{k\in[n]} v_i(A_k)$

- MMS Allocation: A is called MMS if $v_i(A_i) \geq \mu_i$, $\forall i$
- Additive valuations: $v_i(A_i) = \sum_{j\in A_i} v_{ij}$

MMS Value/Partition

Agent\Items	🍎	🍌	🥝
👩🏻	3	1	2
👨🏽	4	4	5

👩🏻	🍎	🍌🥝
Value	3	3
MMS Value	3	

👨🏽	🍎🍌	🥝
Value	8	5
MMS Value	5	

MMS Allocation

Agent\Items	🍎	🍌	🥝
👩🏻	3	1	2
👨🏽	4	4	5

👩🏻	🍎	🍌🥝
Value	3	3
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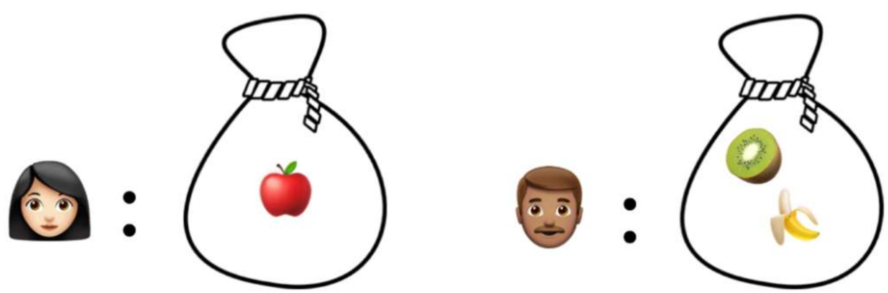

MMS value/partition/allocation

Agent\Items	🍎	🍌	🥝
👩	3	1	2
👨	4	4	5

👩	🍎	🍌 🥝
Value	3	3
MMS Value	3	

👨	🍎 🍌	🥝
Value	8	5
MMS Value	5	

👩 : 🍎　　👨 : 🥝 🍌

Finding MMS value is NP-hard!

What is Known?

- PTAS for finding MMS value [W97]

Existence (MMS allocation)?

- $n = 2$: yes 

  $\Longrightarrow$ A PTAS to find $(1-\epsilon)$-MMS allocation for any $\epsilon > 0$
- $n \geq 3$: NO [PW14]

What is Known?

- PTAS for finding MMS value [W97]

Existence (MMS allocation)?

- $n = 2$: yes 

  $\Longrightarrow$ A PTAS to find $(1-\epsilon)$-MMS allocation for any $\epsilon > 0$
- $n \geq 3$: NO [PW14]

- α-MMS allocation for $\alpha \in [0,1]$: $v_i(A_i) \geq \alpha . \mu_i$
  - $2/3$-MMS exists [PW14, AMNS17, BK17, KPW18, GMT18]
  - $3/4$-MMS exists [GHSSY18]
  - $(3/4 + O(1))$-MMS exists [AG23]
  - $39/40$-MMS does not exist [Feige et al. 2020]

Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$ WHY?

Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$
- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

Properties

- Normalized valuations
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- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

Claim: $V_i(B_i) \geq V_i'(A_i)$

V_i (B_1, B_2) ← (A_1, A_2) V_i'

	~~🍎~~	~~🍌~~	~~🍐~~	~~🍑~~	~~Kiwi~~
👩	3	$\boxed{1}$	2	$\boxed{5}$	$\boxed{4}$
👨	$\boxed{4}$	4	$\boxed{5}$	3	2

→

	1	**2**	**3**	**4**	**5**
👩	5	4	3	2	1
👨	5	4	4	3	2

$V_1(B_1) = 10 > 9$

$V_2(B_2) = 9 > 7$

A B A B A

Picking order

Challenge

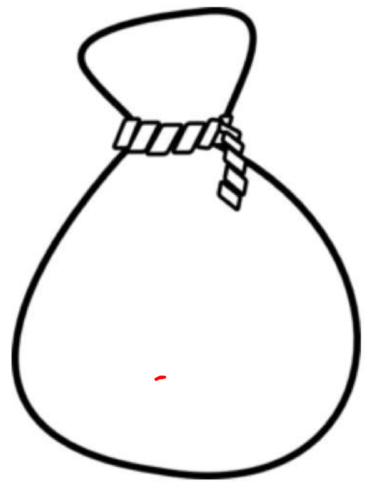
- Allocation of high-value items!
- If for all $i \in N$
  - $v_i(M) = n \Rightarrow \mu_i \leq 1$
  - $\boldsymbol{v_{ij} \leq \epsilon, \forall i, j}$

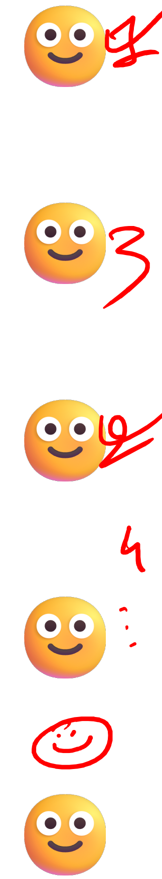
$v_{ij} \leq \epsilon, \forall i, j$

$v_1(B_1) \geq (1-\epsilon) \geq (1-\epsilon)\mu_1$

Claim: After round k, if i remains then v_i(remaining goods) $\geq n - k$.

$k = (n-1)$





Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag B
- Keep adding items to B until some agent i values it $\geq (1-\epsilon)$
- Assign B to i and remove them both

$v_{ij} \leq \epsilon, \forall i, j$

Thm: Every agent gets at least $(1 - \epsilon)$.

Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag B
- Keep adding items to B until some agent i values it $\geq (1 - \epsilon)$
- Assign B to i and remove them both

Warm Up: 1/2-MMS Allocation

- If all $v_{ij} \leq 1/2$ then? $\Leftarrow \varepsilon = 1/2$
  - Done, using bag filling. $\Rightarrow (1-\varepsilon) = 1/2$ - MMS.

- What if some $v_{ij} > \frac{1}{2}$?

Valid Reductions

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$
- Ordered Instance: Agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

- Valid Reduction (α-MMS): If there exists $S \subseteq M$ and $i^* \in N$
  - i^*gets α-MMS value from S $(v_{i^*}(S) \geq \alpha.\mu_{i^*}^n(M))$
  - Once we give S to i^*, and remove both, the MMS value of the remaining agents does not decrease. $\mu_i^{n-1}(M \setminus S) \geq \mu_i^n(M), \forall i \neq i^*$

$\Rightarrow$ reduce the instance size!

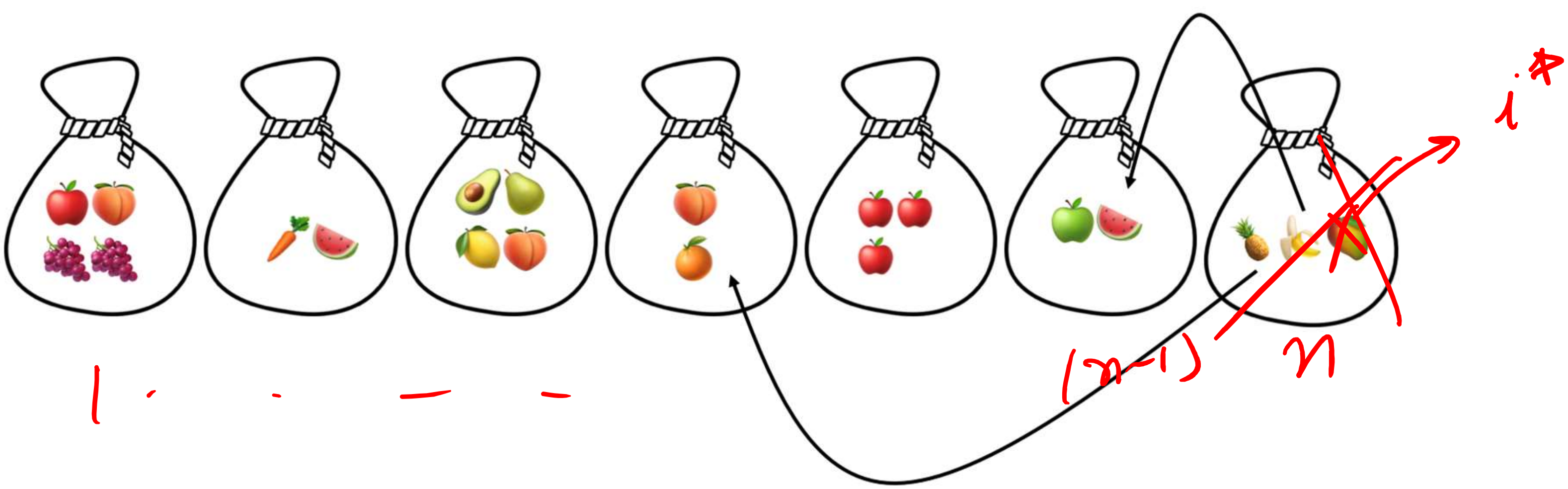
Claim. Suppose agent i $\neq i^*$ gets A_i in an α-MMS allocation of $M \setminus S$ to agents $N \setminus \{i^*\}$, then $(A_1, \dots, A_{i^*-1}, S, A_{i^*+1}, \dots, A_n)$ is an α-MMS allocation in the original instance.

1/2-MMS Allocation

Step 1: Valid Reductions

- If $v_{i^*1} \geq 1/2$ then assign item 1 to i^*

MMS Partition of agent i



1/2-MMS Allocation

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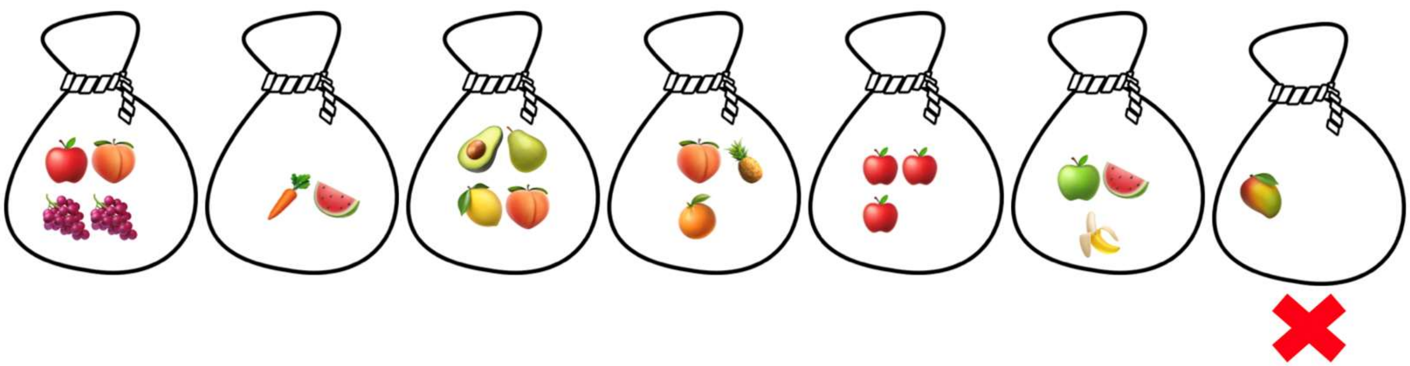


1/2-MMS Allocation

- Re-normalization

Step 0: Normalized Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

Step 1: Valid Reductions

- If $v_{i^*1} \geq 1/2$ then assign item 1 to i^*. Remove good 1 and agent i^*
- After every valid reduction, normalize valuations

Step 2: Bag Filling

$\varepsilon = 1/3 \Leftarrow (1-\varepsilon) =$

2/3-MMS Allocation [GMT19]

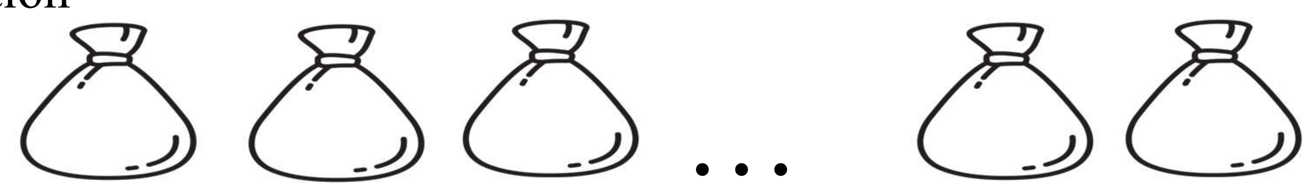
$\le \varepsilon$

- If all $v_{ij} \le 1/3$ then?

Step 1: Valid Reductions

- If $v_{i^*1} \ge 2/3$ then assign item 1 to i^*

MMS Partition of agent i



1 2 3 n-1 n

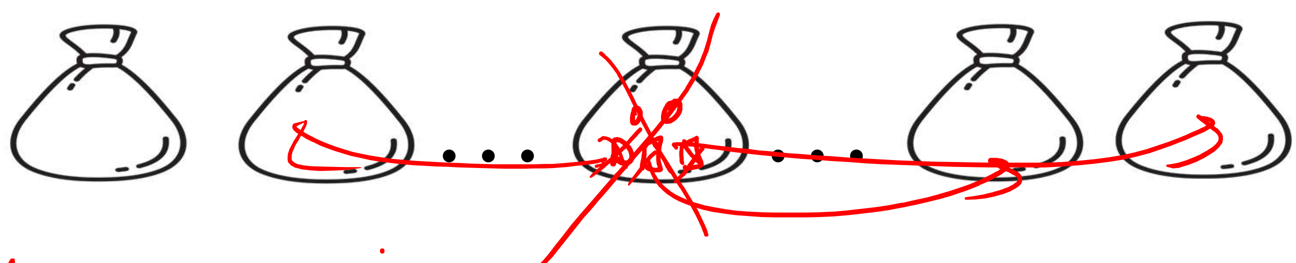
2/3-MMS Allocation [GMT19]

$\forall i: \; v_{i1} \geq v_{i2} \geq \cdots v_{in} \geq v_{i(n+1)} \geq v_{im}.$

Step 1: Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to i^* ✓
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i^*

MMS Partition
of agent i



Case I:
$n, n+1 \; \in A_k$

1 2 A_k n-1 n

2/3-MMS Allocation [GMT19]

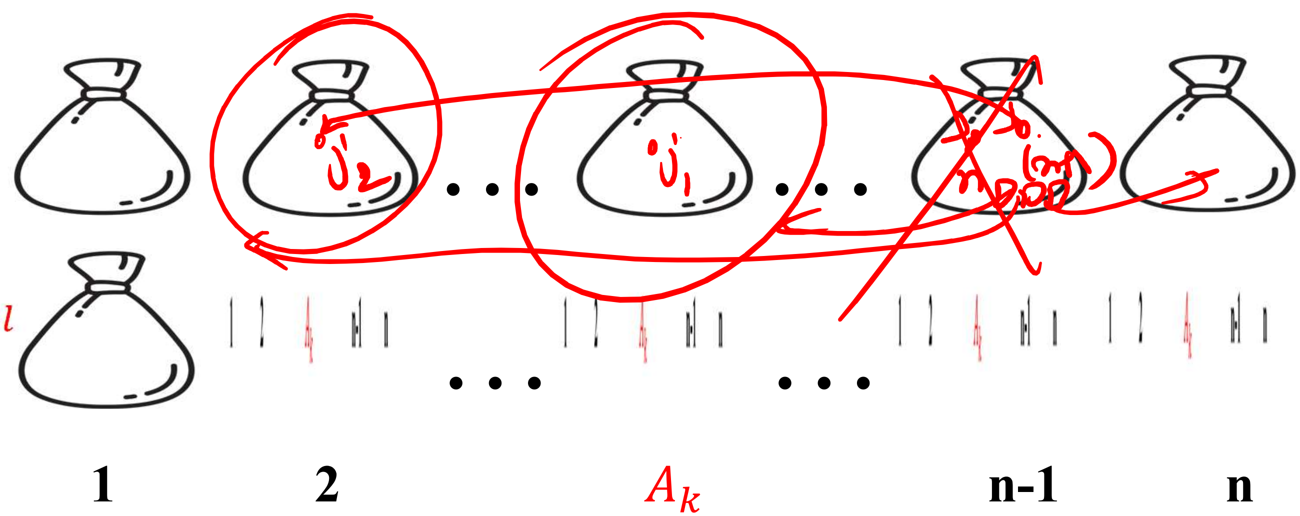
$v_{i1} \geq v_{i2} \geq \cdots v_{ij_1} \geq v_{ij_2} \geq v_{in} \geq v_{i(n+1)}$

Step 1: Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to i^*
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i^*

MMS Partition of agent *i*

Case II:
$n \in A_k$
$(n+1) \in A_l$



$\exists A_d$, with items $j_1 < j_2 \leq (n+1)$. Then, swap items j_1 and n, and items j_2 and (n+1). This may only increase $v_i(A_k)$ & $v_i(A_l)$ because $v_i(j_1) \geq v_i(n)$ & $v_i(j_2) \geq v_i(n+1)$.

2/3-MMS Allocation [GMT19]

$\underbrace{v_{i1} \geq \cdots \geq v_{in} \geq v_{i(n+1)}}_{< 2/3} \geq \cdots v_{im}$

Step 1: Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to i^*
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i^*

MMS Partition
of agent i

Case II:
$n \in A_k$
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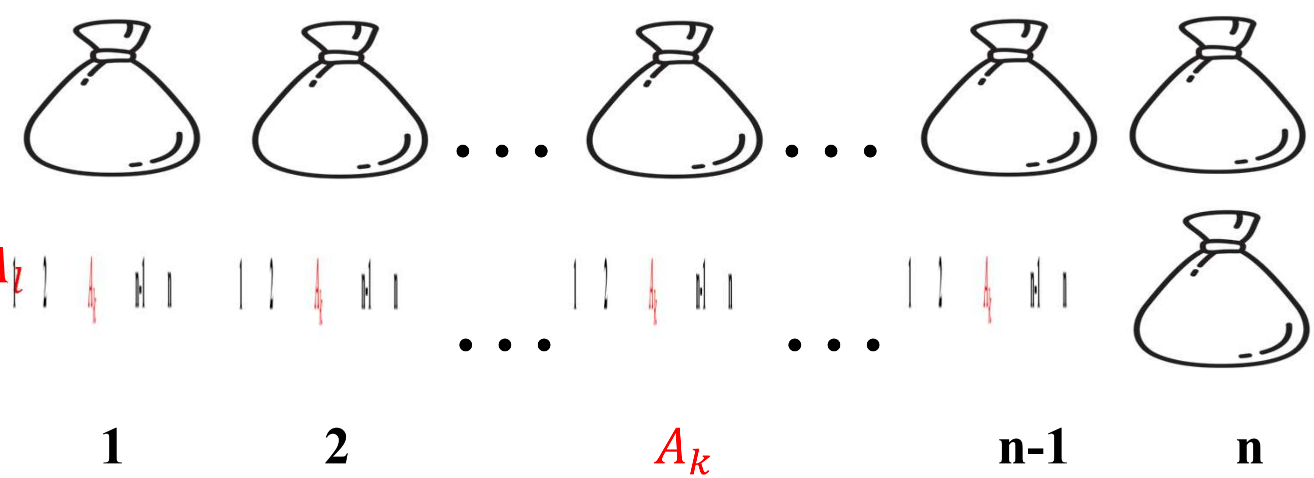

$\exists A_d$, with items $j_1 < j_2 \leq (n+1)$.
Then, swap items j_1 and n, and items j_2 and (n+1).
Move remaining items of A_d to other bundles and remove A_d.

2/3-MMS Allocation [GMT19]

Step 1: Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to i^*
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i^*

MMS Partition
of agent i

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$n \in A_k$
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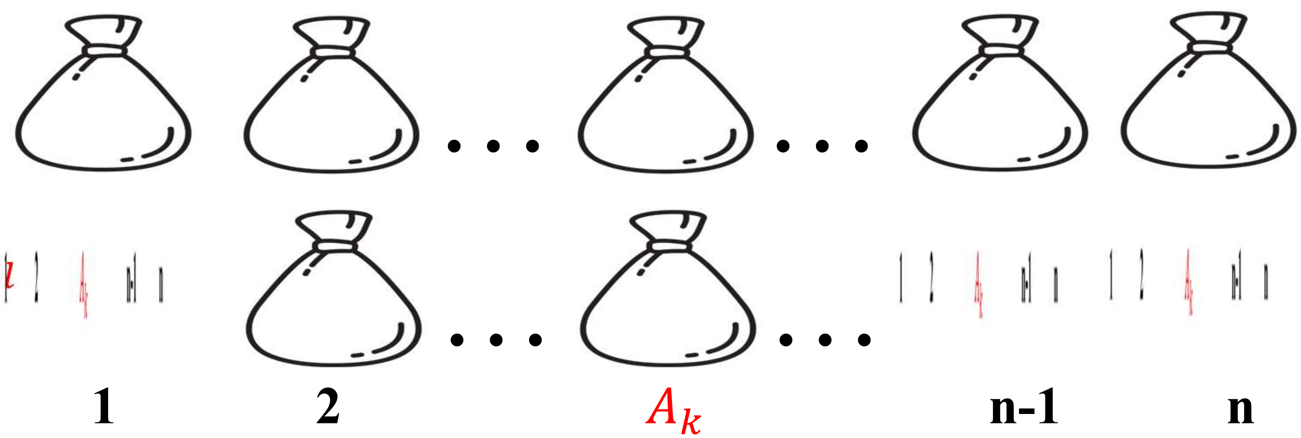


$\exists A_d$, with items $j_1 < j_2 \leq (n+1)$. Then, swap items j_1 and n, and items j_2 and (n+1). Move remaining items of A_d to other bundles and remove A_d.

Again, value of none of the remaining bundles has decreased.
$\Rightarrow$ MMS value of agent i has only increased in the reduced instance.

Step 1: Valid Reductions

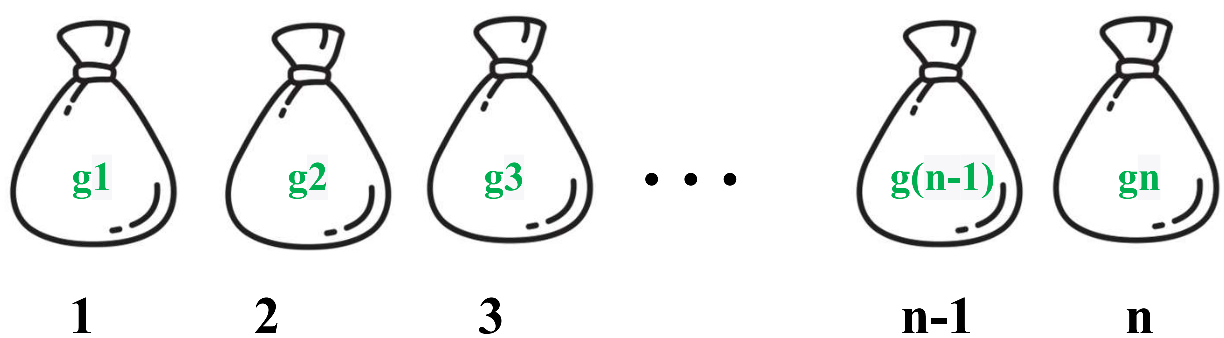
- If $v_{i^*1} \geq 2/3$ then assign item 1 to i^*
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i^*

Step 2: Generalized Bag Filling with $\epsilon = \frac{1}{3}$

- Initialize n bags $\{B_1, \dots B_n\}$ with $B_k = \{k\}, \forall k$.
- Assign items starting from $(n+1)$th to the first available bag, and give it to the first agent who shouts (values it at least $2/3 = (1-\epsilon)$).

After Step 1,
For each agent i,
$v_{ij} < \frac{2}{3}, \forall j \leq n$
$v_{ij} < \frac{1}{3}, \forall j > n$

Claim. If agent i^* is the first to shout, then for any agent $i \neq i^*$ the bag is of value at most 1.



2/3-MMS Allocation [GMT19]

- (Re)normalization

Step 0: Normalized Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

Step 1: Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to i^*
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to i^*
- After every valid reduction, normalize valuations

Step 2: Generalized Bag Filling with $\epsilon = \frac{1}{3}$

- Initialize n bags $\{B_1, \ldots B_n\}$ with $B_k = \{k\}, \forall k$

Chores

- N: set of n agents, 1 ,…, n,
- M: set of m indivisible chores

- Agent i has a disutility function $d_i : 2^m \to \mathbb{R}_+$ over subsets of items
  - Monotone: the more the **un**-happier
- Additive: $d_i(S) = \sum_{j \in S} d_{ij}$, for any subset $S \subseteq M$

- N: set of n agents, 1 ,…, n,
- M: set of m indivisible chores
- Agent i has a disutility function $d_i : 2^m \to \mathbb{R}_-$ over subsets of items
  - Additive: $d_i(S) = \sum_{j \in S} d_{ij}$, for any subset $S \subseteq M$

$$\text{Allocation } A = (A_1, \ldots, A_n)$$

EF1: No agent envies another after removing one of her chores.

$$\forall i, k \in N, \qquad d_i(A_i \backslash c) \le d_i(A_k), \qquad \exists c \in A_i$$

EF1: Algorithms

Round Robin

1. Order agents arbitrarily.
2. Let them pick their best chore (least painful chore), one-at-a-time, in that order.

Observations:

- If agent k picks the last chore, then agent $(k+1)$ does not envy anyone. Why?

EF1: Algorithms

Envy-cycle-elimination

1. $A = (\emptyset, \ldots, \emptyset)$
2. While there are unassigned chores
   1. Construct envy-graph of A and remove any cycles.
   2. Give an unassigned chore to ….. ??

Observations:

- Cycle elimination does not increase any agent's disutility.
- Giving a chore to sink maintains EF1. Why?

MMS

- N: set of n agents, 1 ,…, n,
- M: set of m indivisible chores
- Agent i has a disutility function $d_i : 2^m \to \mathbb{R}_-$ over subsets of items
  - Additive: $d_i(S) = \sum_{j \in S} d_{ij}$, for any subset $S \subseteq M$

- $\Pi :=$ Set of all partitions of items into n bundles

MMS value: $\mathrm{MMS_i} = \mu_i = \min_{A \in \Pi} \max_{A_k \in A} d_i(A_k)$

α-MMS allocation for $\alpha \geq 1$: $\forall i,\ d_i(A_i) \leq \alpha \mu_i$

1-MMS allocation may not exist!

EF1 to α-MMS

Claim. If $(A_1, \ldots, An)$ is EF1 then it is 2-MMS

Observations: $\mu_i \geq \frac{d_i(M)}{n}$ and $\mu_i \geq \max_{j \in M} d_{ij}$

Proof.

Summary

Covered

- Additive Valuations:
  - ½-MMS allocation (poly-time algorithm)
  - 2/3-MMS allocation (polynomial-time algorithm)

State-of-the-art

- $\left(\frac{3}{4}+\right)$-MMS allocation [AGT23]
- More general valuations
  - MMS [GHSSY18]
- Groupwise-MMS [BBKN18]
- Chores: 11/9-MMS [HL19]

Major Open Questions (additive)

- c-MMS + PO: polynomial-time algorithm for a constant $c > 0$
- Existence of 4/5-MMS allocation? For 5 agents?

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