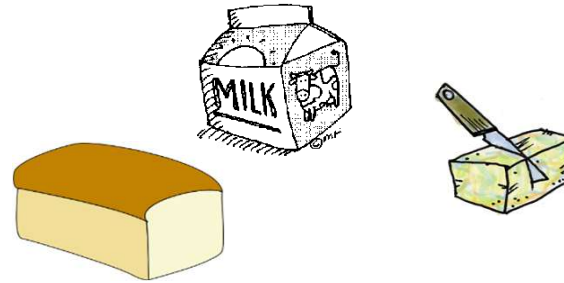


CS 580: Topics on AGT

Lec 2: Fair Division of Divisibles via Competitive Equilibrium

Instructor: Ruta Mehta

Divisible goods



Goal: Find *fair* and *efficient* allocation



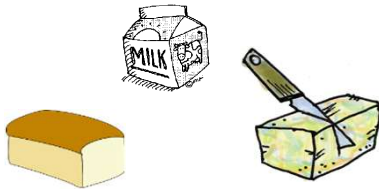
1.



Model

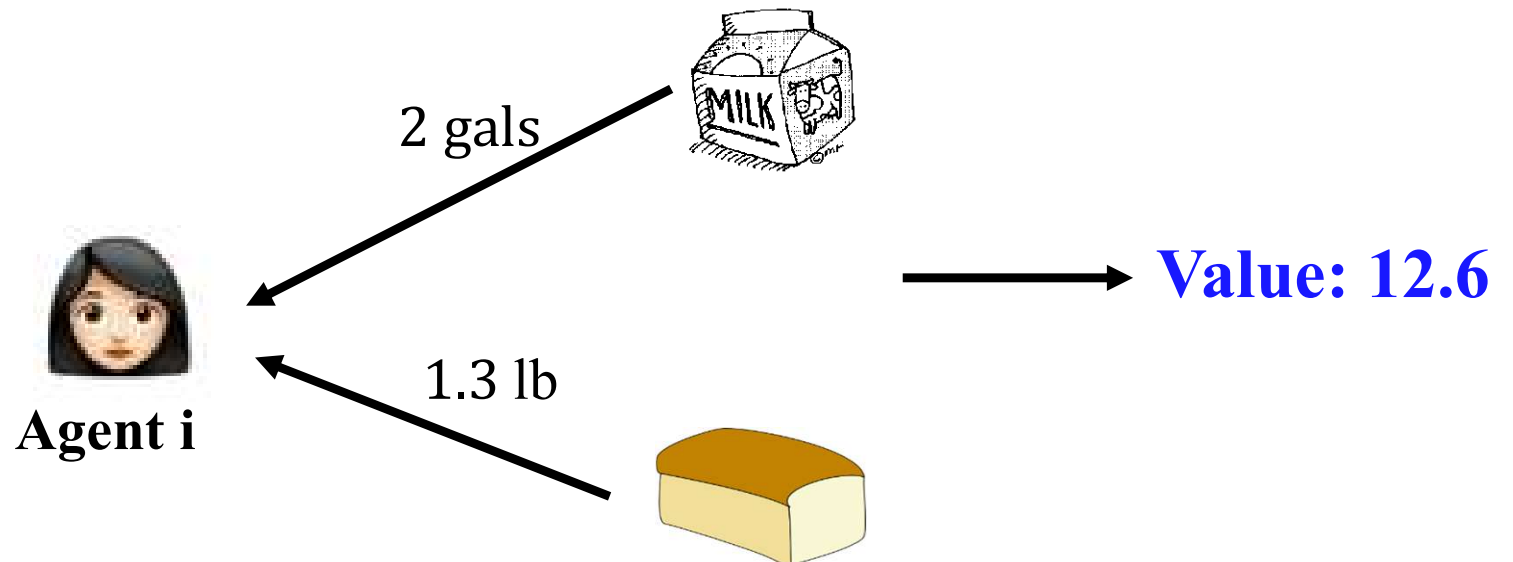


- A : set of n agents
- M : set of m **divisible** goods (manna)



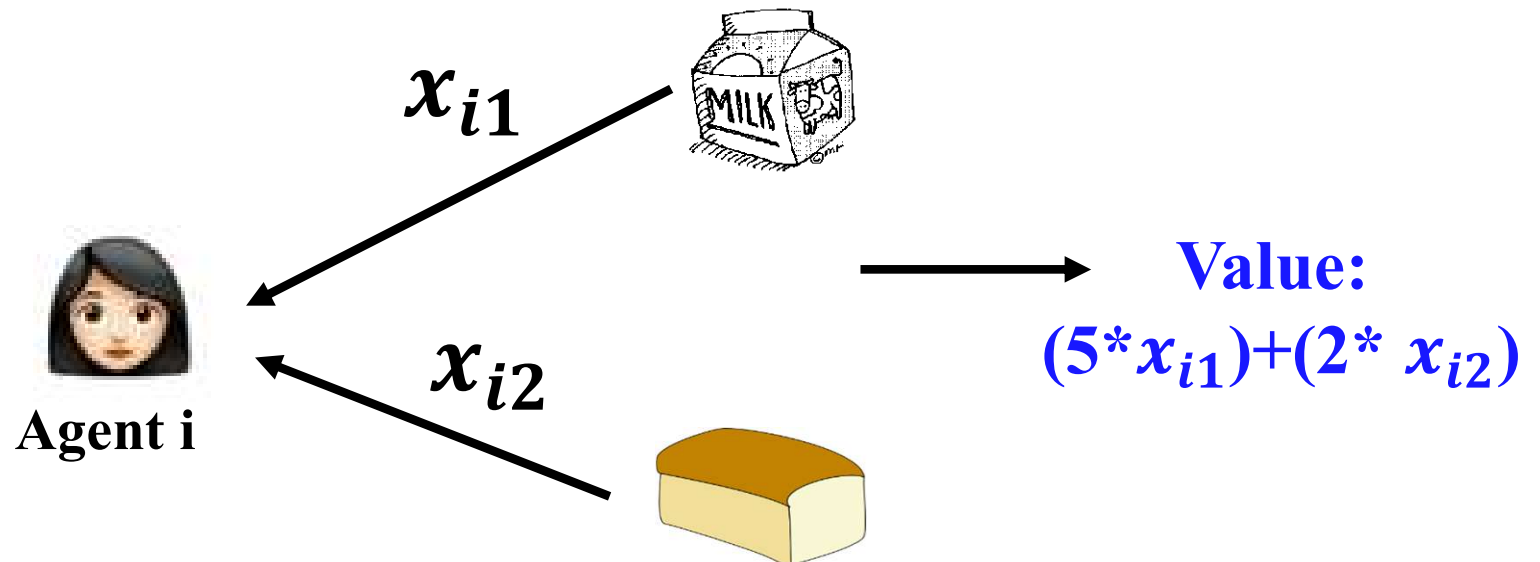
- Each agent i has
 - Valuation function $V_i: R_+^m \rightarrow R_+$ over bundles of items

Valuation function



Values milk at 5/gallon, and bread at 2/lb

Valuation function



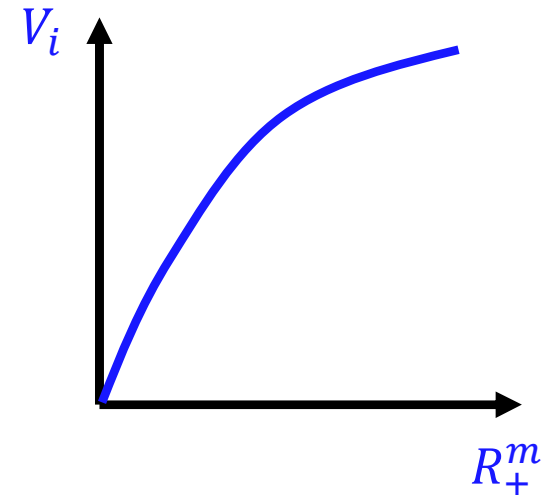
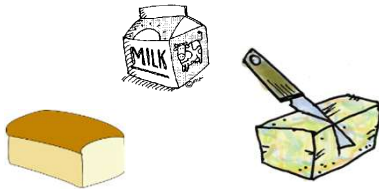
Values milk at 5/gallon, and bread at 2/lb

Linear/Additive Valuation

Model



- A : set of n agents
- M : set of m **divisible** goods (manna)



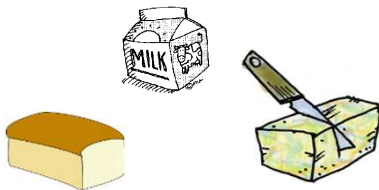
- Each agent i has
 - Valuation function $V_i: R_+^m \rightarrow R_+$ over bundles of items
 - Concave: Captures *decreasing marginal returns*.

Goal: Find *fair* and *efficient* allocation

Model

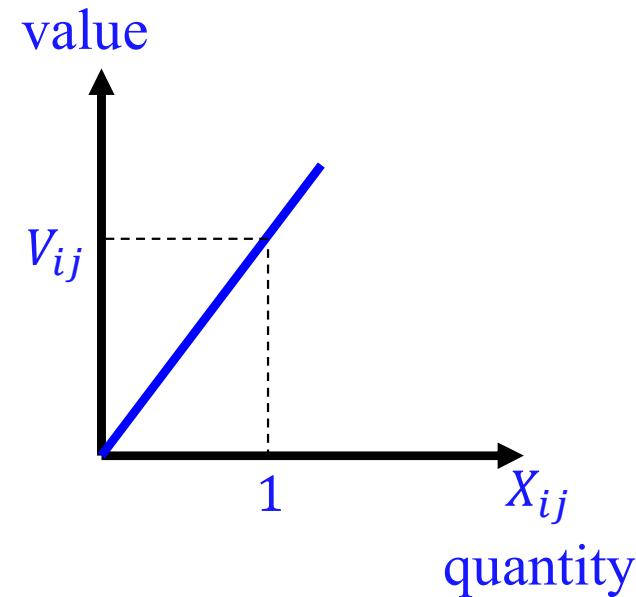


- A : set of n agents
- M : set of m **divisible** goods (manna)



- Each agent i has
 - Additive/linear $V_i: R_+^m \rightarrow R_+$:

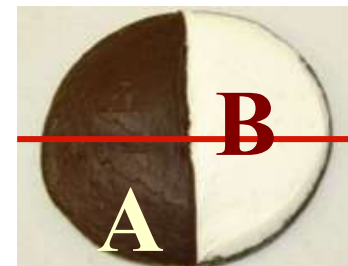
$$V_i(X_{i1}, \dots, X_{im}) = \sum_{j \in M} v_{ij} X_{ij}$$



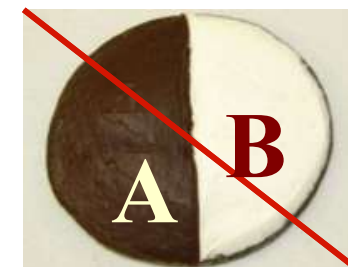
Example: Half moon cookie



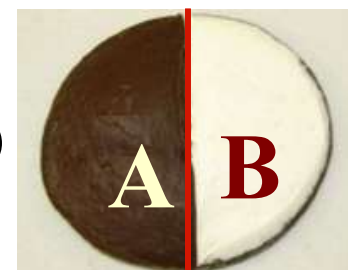
(i)



(ii)



(iii)



Fair (Agreeable)

Efficient (Non-wasteful)

Allocation: Bundle $X_i \in R_+^m$ to agent i

Envy-free: No agent *envies* other's allocation over her own.

For each agent i ,
$$V_i(X_i) \geq V_i(X_j), \forall j \in [n]$$

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

For each agent i , $V_i(X_i) \geq \frac{V_i(M)}{n}$

Pareto-optimal: No other allocation is better for all.

There is no Y , s. t.
$$V_i(Y_i) \geq V_i(X_i), \forall i \in [n]$$

Welfare Maximizing
(max: $\sum_i V_i$)

Fair (Agreeable)

Efficient (Non-wasteful)

Envy-free: No agent *envies* other's allocation over her own.

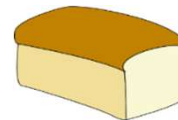
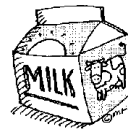
Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

**Allocation
in red**

[3, 2, 2]
[0, 0, 0]



[20, 20, 30]
[0, 0, 0]



Fair (Agreeable)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

**Allocation
in red**

[3, 2, 2]
[1/2, 1/2, 1/2]

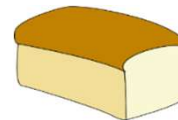
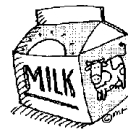


[20, 20, 30]
[1/2, 1/2, 1/2]



Efficient (Non-wasteful)

Pareto-optimal: No other allocation is better for all.



Fair (Agreeable)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

**Allocation
in red**

[3, 2, 2]
[1, 1/2, 0]



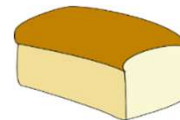
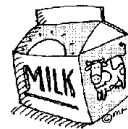
[20, 20, 30]
[0, 1/2, 1]



Efficient (Non-wasteful)

Pareto-optimal: No other allocation is better for all.

Welfare Maximizing
($\max: \sum_i V_i$)



Fair (Agreeable)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

**Allocation
in red**

[3, 2, 2]
[0, 0, 0]



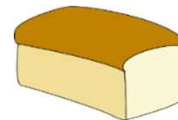
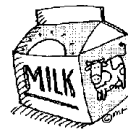
[20, 20, 30]
[1, 1, 1]



Efficient (Non-wasteful)

Pareto-optimal: No other allocation is better for all.

Welfare Maximizing
($\max: \sum_i V_i$)



Fair (Agreeable)

Envy-free: No agent *envies* other's allocation over her own.

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

**Allocation
in red**

[3, 2, 2]
[1, 1/2, 0]



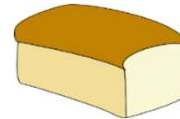
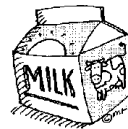
[20, 20, 30]
[0, 1/2, 1]



Efficient (Non-wasteful)

Pareto-optimal: No other allocation is better for all.

**(Nash) Welfare
Maximizing $(\prod_i V_i)$**



Fair (Agreeable)

**Efficient
(Non-wasteful)**

Envy-free

Pareto-optimal

Proportional

**(Nash) Welfare
Maximizing**

**Competitive Equilibrium
(with equal income)**

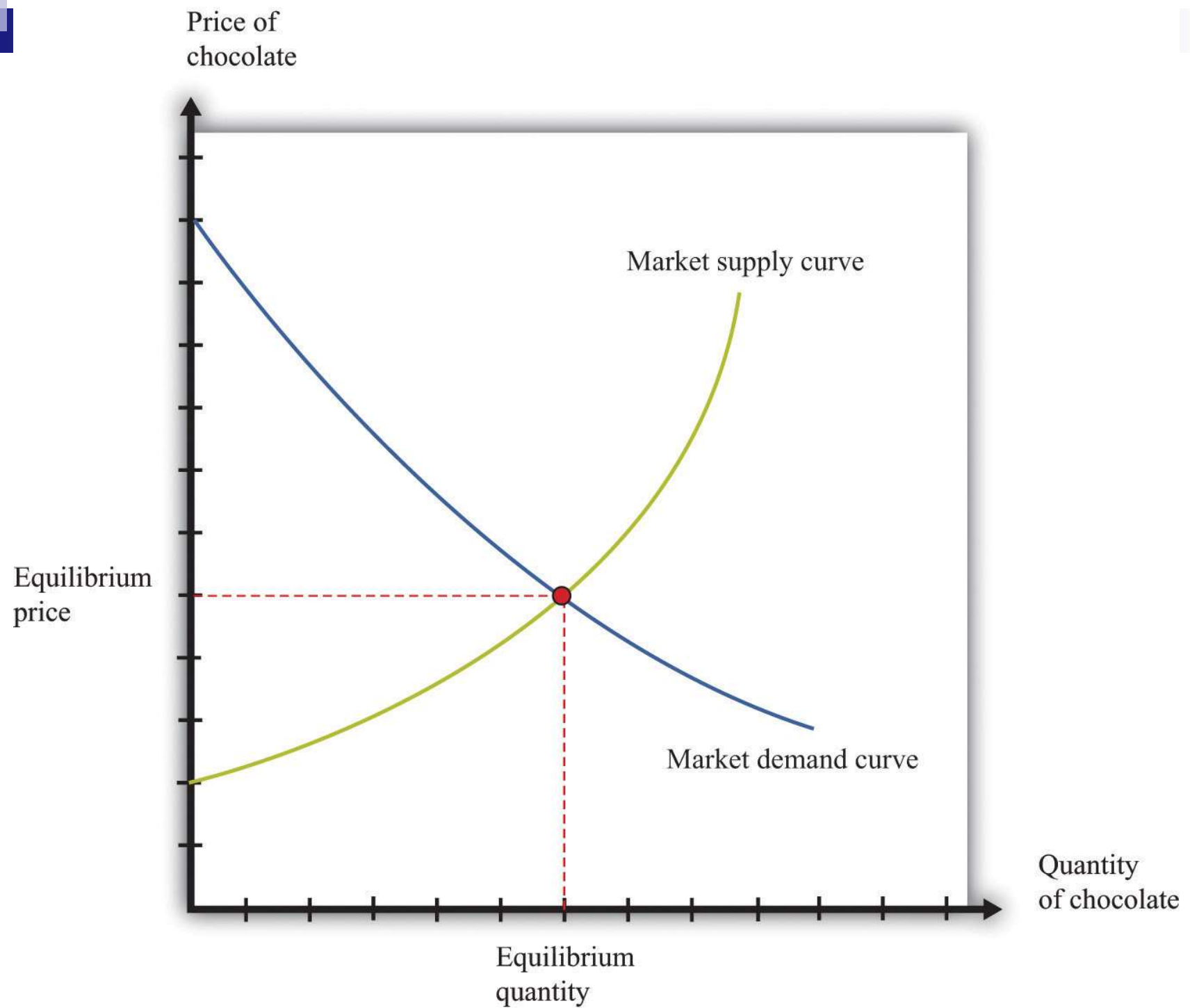
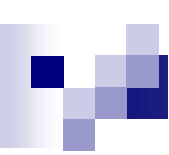
Beginning of Competitive Equilibrium



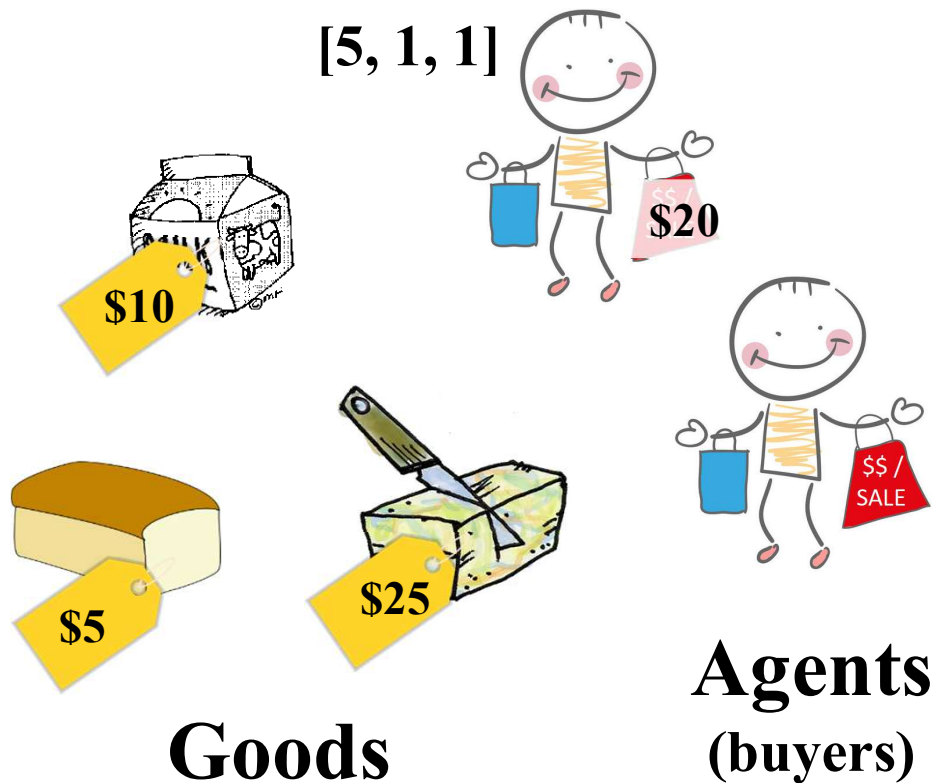
Adam Smith
(1776)

Invisible hand

“Economic concept that describes the unintended greater social benefits and public good brought about by individuals acting in their own self-interests.^{[1][2]} The concept was first introduced by Adam Smith in *The Theory of Moral Sentiments*, written in 1759. According to Smith, it is literally divine providence, that is the hand of God, that works to make this happen.”



Competitive (market) Equilibrium (CE)



Demand optimal bundle
 $\operatorname{argmax}_{\{X \text{ affordable}\}} V_i(X)$

Competitive (market) Equilibrium (CE)



CE Example

Demands

[2, 0]

[5, 1]



[1, 4]



[0, 1]

Supply



1



1

**Total
Demand**

2 > 1

Demand > Supply

1

CE Example



Demand

1

**Demand = Supply
⇒ CE**



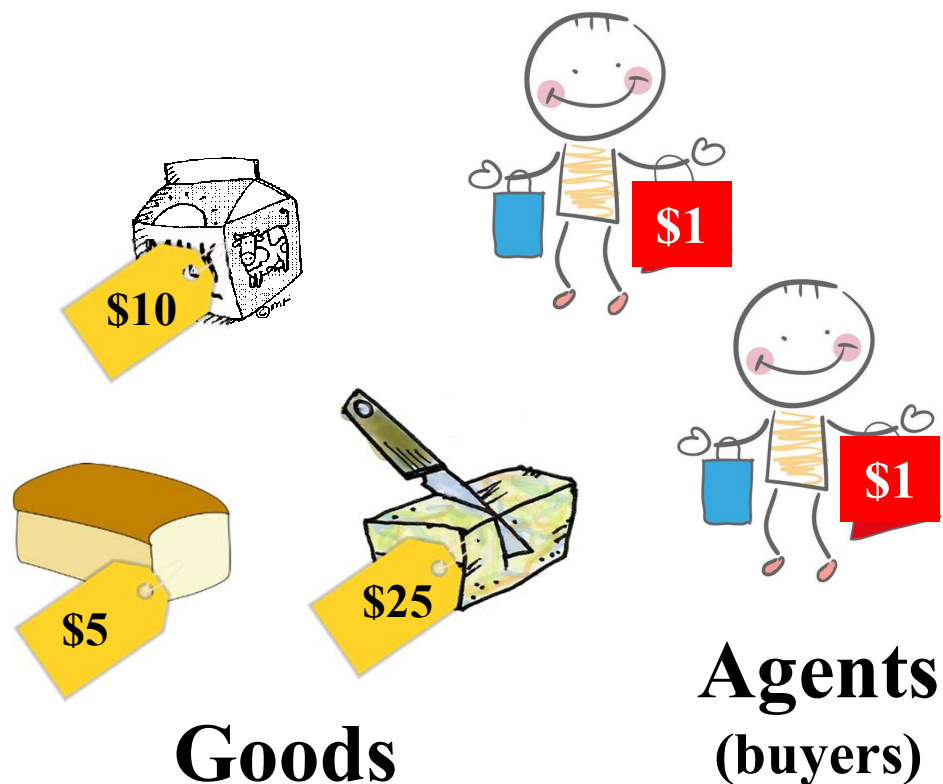
1

$[0, 1]$

w/ equal income (CEEI):

Agents have the same amount of money

CEEI: Properties



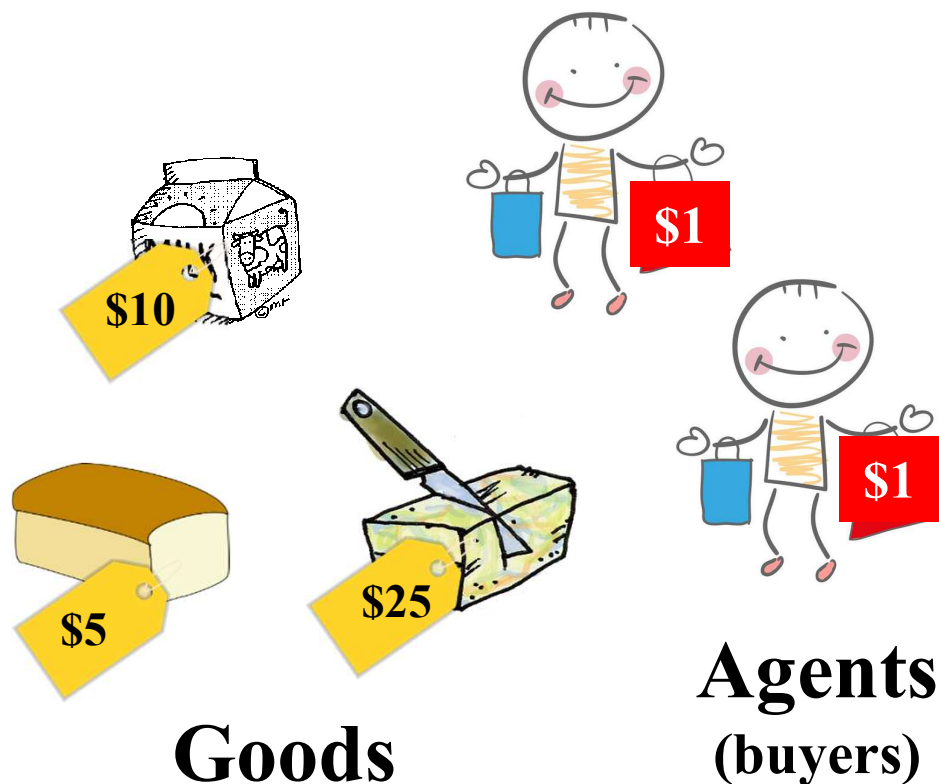
An agent can afford anyone else's bundle, but demands her own
 $\Rightarrow$ **Envy-free**

1^{st} welfare theorem
 $\Rightarrow$ **Pareto-optimal**

Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

CEEI: Properties



Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

**Envy-free & “Demand=Supply”
⇒ Proportional**

Proof.

Envyfree

$$\Rightarrow V_i(\bar{X}_i) \geq V_i(\bar{X}_k), \forall k \in [n]$$

$$\Rightarrow nV_i(\bar{X}_i) \geq \sum_{k \in [n]} V_i(\bar{X}_k)$$

“Demand = Supply”

$$\Rightarrow \sum_{k \in [n]} V_i(\bar{X}_k) \geq V_i(M) (\because V_i \text{ concave})$$

$$\Rightarrow V_i(\bar{X}_i) \geq \frac{V_i(M)}{n}$$

CE History



**Adam Smith
(1776)**



**Leon Walras
(1880s)**



Irving Fisher (1891)



**Arrow-Debreu (1954)
(Nobel prize)**

(Existence of CE in the
exchange model w/ firms)

...

Computation of CE (w/ goods)

Algorithms

- Convex programming formulations
 - Eisenberg-Gale (1959): CEEI w/ 1-homogeneous valuations
 - Shmyrev (2009), DGV (2013), CDGJMVY (2017) ...
- (Strongly) Poly-time algorithms (linear valuations)
 - DPSV (2002), Orlin (2010), DM (2015), GV (2019) ...
- Simplex-like algorithms: Eaves (1976), GM.SV (2011), GM.V (2014), ...

Complexity

- PPAD: Papadimitrou'92, CDDT'09, VY'11, CPY'17, Rubinstein'18, ...
- FIXP: EY'09, GM.VY'17, F-RHHH'21 ...

Learning: RZ'12, BDM.UV'14, ..., FPR'22, ...

Matching/mechanisms: BLNPL'14, ..., KKT'15, ..., FGL'16, ..., AJT'17, ..., BGH'19, BNT-C'19, ...

*Alaei, Bei, Branzei, Chen, Cole, Daskalakis, Deng, Devanur, Duan, Dai, Etessami, Feldman, Fiat, Filos-Ratsikas, Garg, Gkatzelis, Hansen, Hogh, Hollender, Jain, Jalaly, Hoefer, Kleinberg, Lucier, Mai, Mehlhorn, Mehta, Mansour, Morgenstern, Nisan, Paes, Lee, Leme, Papadimitriou, Paparas, Parkes, Roth, Saberi, Sohoni, Talgam-Cohen, Tardos, Vazirani, Vegh, Yazdanbod, Yannakakis, Zhang,... ...

Simple Tatonnement Procedure (Algo)

Increase prices of the over demanded goods.

Theorem. Tatonnement process Converges to a CE if V_i s are *weak gross substitutes (WGS)*.

WGS: Increase in price of a good does not decrease demand of any other good.

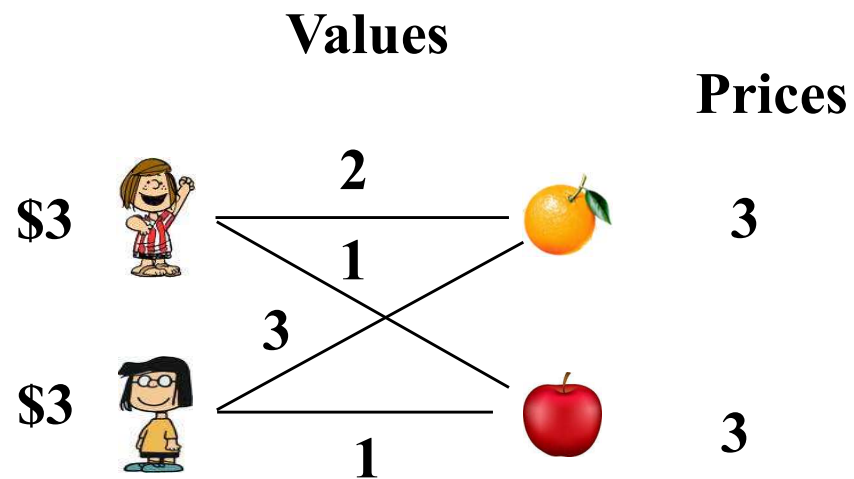
Example: Linear V_i s

$$V_i(X_i) = \sum_{j \in [m]} V_{ij} X_{ij}$$

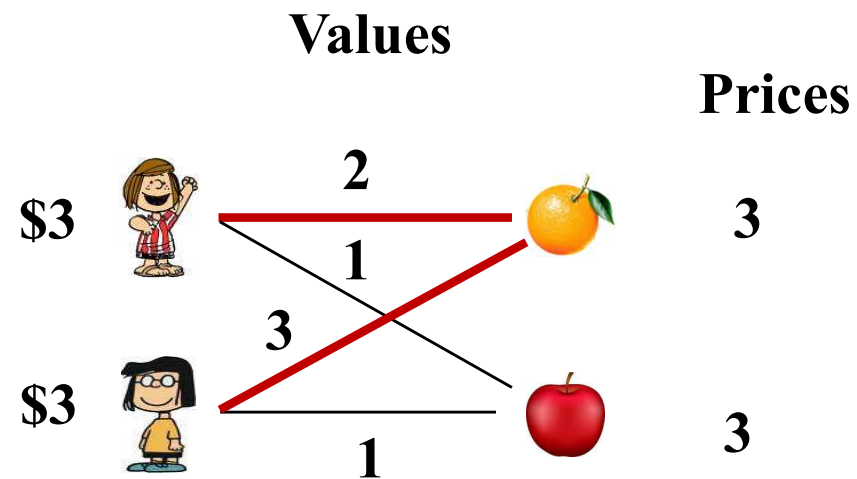


Fast Computation: Characterization

Example (Intuition)

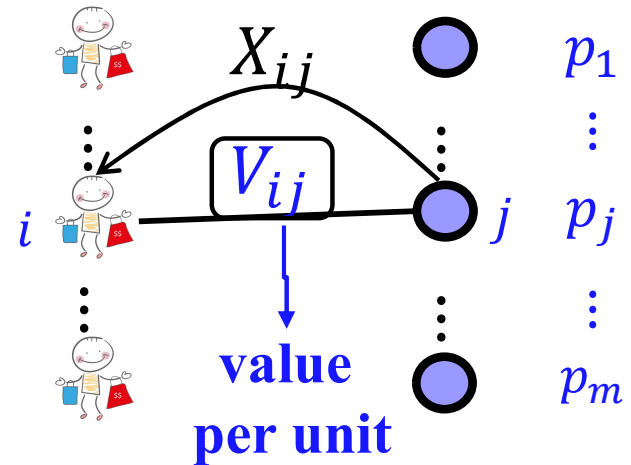


Example (Intuition)



Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



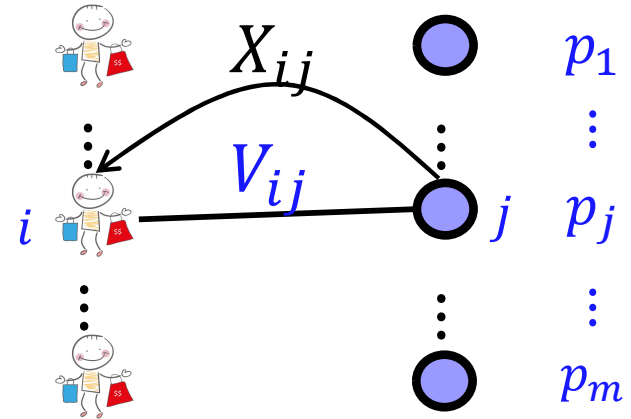
Optimal bundle: can spend at most *one* dollar.

Must spend wisely: on goods that give maximum value-per-dollar $\frac{V_{ij}}{p_j}$

Must spend all of it: $\sum_{j \in M} p_j X_{ij} = 1$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$

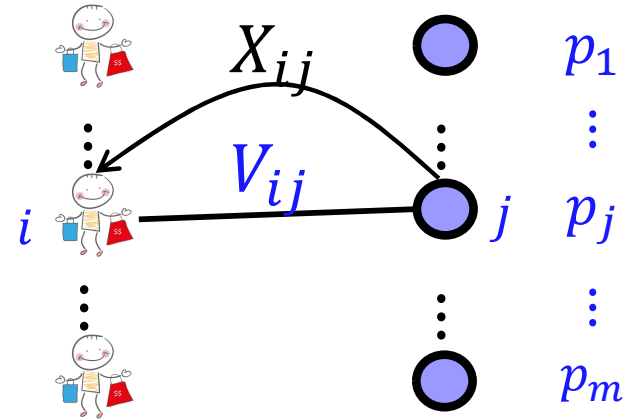


Optimal bundle: can spend at most *one* dollar.

$$\sum_{j \in M} V_{ij} X_{ij} = \sum_j \underbrace{\frac{V_{ij}}{p_j}}_{\text{value per dollar spent (bang-per-buck)}} \underbrace{(p_j X_{ij})}_{\text{(\$ spent)}} \leq \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) \sum_j p_j X_{ij} \leq \underbrace{\left(\max_{k \in G} \frac{V_{ik}}{p_k} \right)}_{\text{MBB Maximum bang-per-buck}} 1$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollar.

$$\sum_{j \in M} V_{ij} X_{ij} = \sum_j \boxed{\frac{V_{ij}}{p_j}} (p_j X_{ij}) \leq \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) \sum_j p_j X_{ij} \leq \boxed{\left(\max_{k \in G} \frac{V_{ik}}{p_k} \right)} 1$$

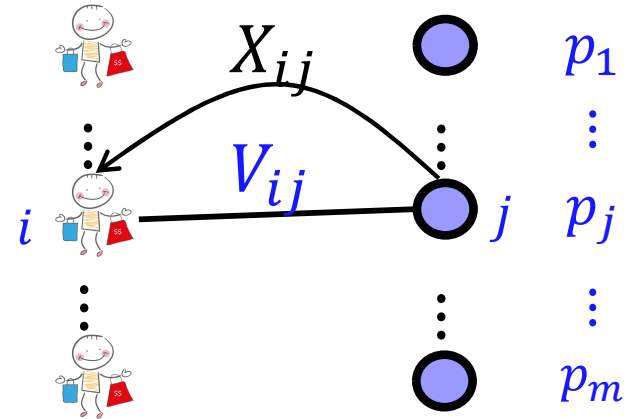
value per dollar spent (bang-per-buck) iff MBB
Maximum bang-per-buck

Buy only MBB goods.

$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollar.

$$\sum_{j \in M} V_{ij} X_{ij} = \sum_j \underbrace{\frac{V_{ij}}{p_j}}_{\text{value per dollar spent (bang-per-buck)}} (p_j X_{ij}) \stackrel{\text{iff}}{=} \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) \sum_j p_j X_{ij} \stackrel{\text{iff}}{=} \underbrace{\left(\max_{k \in G} \frac{V_{ik}}{p_k} \right)}_{\text{MBB Maximum bang-per-buck}} 1$$

Buy only MBB goods.

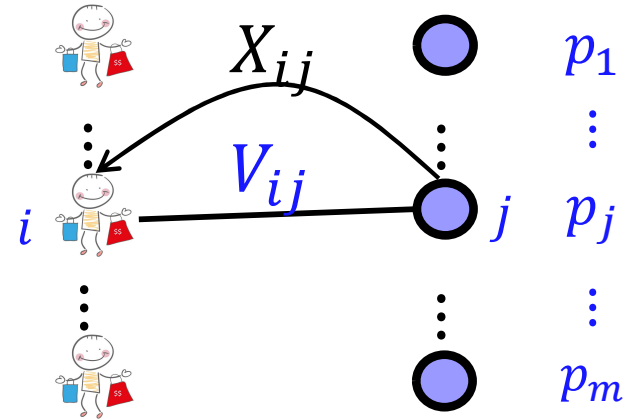
$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

Spends all of 1 dollar.

$$\sum_j p_j X_{ij} = 1$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollars.

$$\sum_{j \in M} V_{ij} X_{ij} \leq \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) \mathbf{1}$$

iff

1. Buy only MBB goods.

$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

2. Spends all of 1 dollar.

$$\sum_j p_j X_{ij} = 1$$

Linear V_i s: CEEI Characterization

Prices $p = (p_1, \dots, p_m)$ and allocation $X = (\bar{X}_1, \dots, \bar{X}_n)$ are at equilibrium iff

■ Optimal bundle (OB): For each agent i

$$\square \sum_j p_j X_{ij} = 1$$

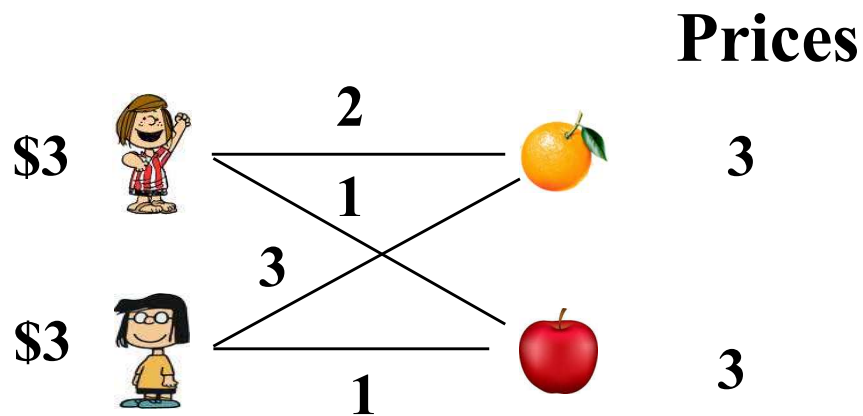
$$\square X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = \max_{k \in M} \frac{V_{ik}}{p_k}, \text{ for all good } j$$

■ Market clears: For each good j ,

$$\sum_i X_{ij} = 1.$$

Example

- 2 Buyers ( , ), 2 Items ( , ) with unit supply
- Each buyer has budget of \$3 and a linear utility function



Example

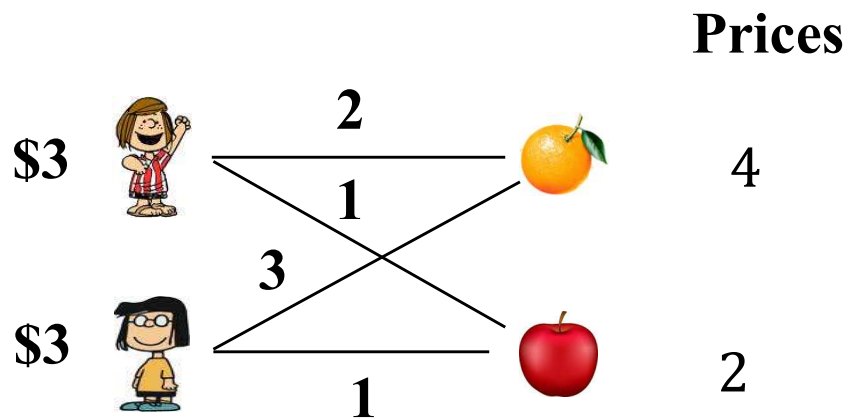
- 2 Buyers ( , ), 2 Items ( , ) with unit supply
- Each buyer has budget of \$3 and a linear utility function



Not an Equilibrium!

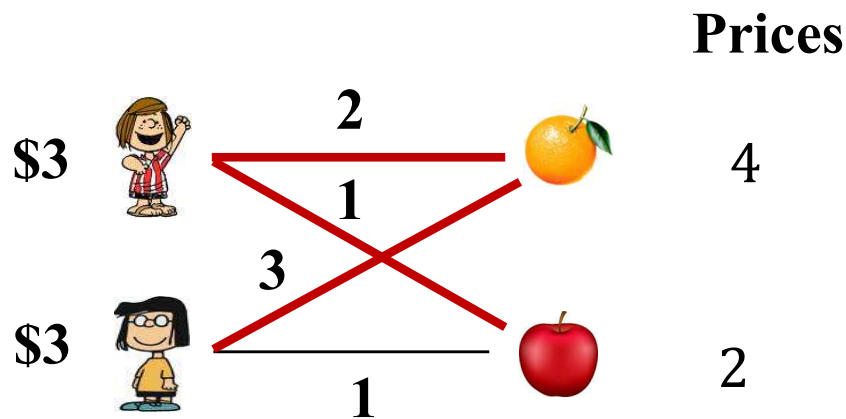
Example

- 2 Buyers ( , ), 2 Items ( , ) with unit supply
- Each buyer has budget of \$3 and a linear utility function



Example

- 2 Buyers (👧, 👦), 2 Items (🍊, 🍎) with unit supply
- Each buyer has budget of \$3 and a linear utility function



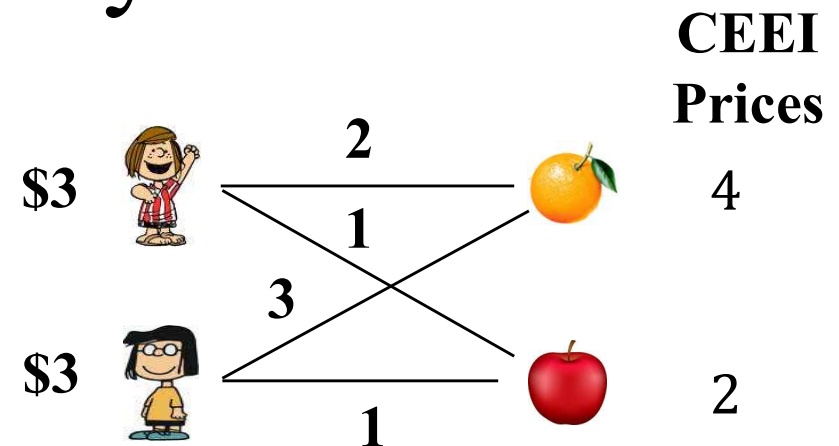
Demand = Supply

Equilibrium!

CEEI Properties: Summary

CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional



CEEI Properties: Summary

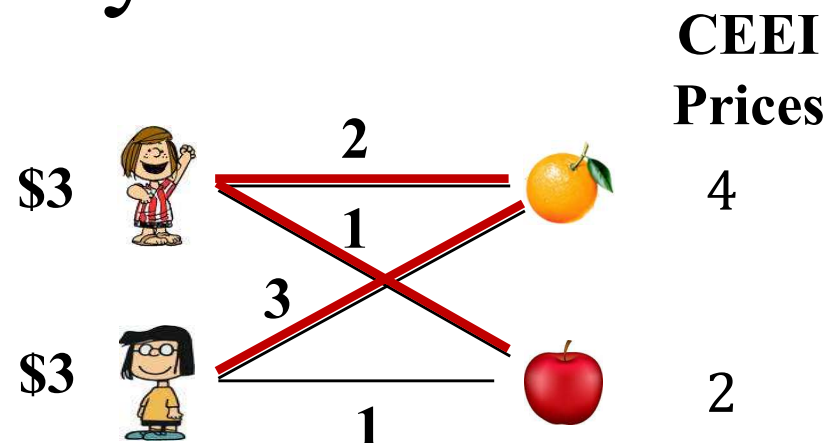
CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional

$$\forall i: v_i(x_i) \geq \frac{v_i(M)}{n}$$

Next...

- **Nash welfare maximizing**



CEEI Allocation:

$$X_1 = \left(\frac{1}{4}, 1\right), X_2 = \left(\frac{3}{4}, 0\right)$$

$$V_1(X_1) = \frac{3}{2}, V_2(X_2) = \frac{9}{4}$$

$$V_1(X_2) = \frac{3}{2}, V_2(X_1) = \frac{7}{4}$$



Social Welfare

$$\sum_{i \in A} V_i(X_{i1}, \dots, X_{im})$$

Utilitarian

Issues: May assign 0 value to some agents.
Not scale invariant!

Max Nash Welfare

$$\mathbf{max:} \prod_{i \in A} V_i(X_{i1}, \dots, X_{im})$$

$$\begin{aligned} \mathbf{s.t.} \quad & \sum_{i \in A} X_{ij} \leq 1, \quad \forall j \in G \\ & X_{ij} \geq 0, \quad \forall i, \forall j \end{aligned}$$

Feasible allocations

Max Nash Welfare (MNW)

$$\mathbf{max:} \log \left(\prod_{i \in A} V_i(X_{i1}, \dots, X_{im}) \right)$$

$$\begin{aligned} \mathbf{s.t.} \quad & \sum_{i \in A} X_{ij} \leq 1, \quad \forall j \in G \\ & X_{ij} \geq 0, \quad \forall i, \forall j \end{aligned}$$

Feasible allocations

Max Nash Welfare (MNW)

$$\text{max: } \sum_{i \in A} \log V_i(X_{i1}, \dots, X_{im})$$

$$\begin{aligned} \text{s.t. } \quad & \sum_{i \in A} X_{ij} \leq 1, \quad \forall j \in G \\ & X_{ij} \geq 0, \quad \forall i, \forall j \end{aligned}$$

Feasible allocations

Eisenberg-Gale Convex Program '59

$$\mathbf{max:} \quad \sum_{i \in A} \log V_i(\bar{X}_i)$$

Dual var.

$$\mathbf{s.t.} \quad \sum_{i \in A} X_{ij} \leq 1, \quad \forall j \in G \longrightarrow p_j$$
$$X_{ij} \geq 0, \quad \forall i, \forall j$$

Theorem. Solutions of EG convex program are exactly the CEEI (p, X) .

Proof.

Consequences: CEEI

- Exists
- Forms a convex set
- Can be *computed* in polynomial time
- Maximizes Nash Welfare

Theorem. Solutions of EG convex program are exactly the CEEI (p, X) .

Proof. $\Rightarrow$ (Using KKT)

Recall: CEEI Characterization

Prices $p = (p_1, \dots, p_m)$ and allocation $X = (X_1, \dots, X_n)$

■ **Optimal bundle:** For each buyer i

□ $p \cdot X_i = 1$

□ $X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = \max_{k \in M} \frac{V_{ik}}{p_k}, \text{ for all good } j$

■ **Market clears:** For each good j ,

$$\sum_i X_{ij} = 1.$$

Theorem. Solutions of EG convex program are exactly the CEE.

Proof. $\Rightarrow$ (Using KKT)

$$\forall j, p_j > 0 \Rightarrow \sum_i X_{ij} = 1$$

$$\begin{array}{ll} \max: & \sum_{i \in A} \log(V_i(\bar{X}_i)) \xrightarrow{\sum_j V_{ij} X_{ij}} \\ \text{s.t.} & \sum_{i \in A} X_{ij} \leq 1, \forall j \in G \longrightarrow p_j \geq 0 \\ & X_{ij} \geq 0, \quad \forall i, \forall j \end{array}$$

Dual var.

Dual condition to X_{ij} :

$$\frac{V_{ij}}{V_i(X_i)} \leq p_j \Rightarrow \frac{V_{ij}}{p_j} \leq V_i(X_i) \Rightarrow p_j > 0 \Rightarrow \text{market clears}$$

buy only MBB goods

$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = V_i(X_i)$$

$$\begin{aligned} \sum_j V_{ij} X_{ij} &= (\sum_j p_j X_{ij}) V_i(X_i) \\ &\Rightarrow \sum_j p_j X_{ij} = 1 \end{aligned}$$

$\Rightarrow$ optimal bundle

Efficient (Combinatorial) Algorithms

Polynomial time

- Flow based [DPSV'08]
 - General exchange model (barter system) [DM'16, DGM'17, CM'18]
- Scaling + Simplex-like path following [GM.SV'13]

Strongly polynomial time

- Scaling + flow [O'10, V'12]
 - Exchange model (barter system) [GV'19]

We will discuss some of these if there is interest.

Application to Display Ads: Pacing Eq.

- Google Display Ads

- Each advertiser has

- Budget B_i . Value v_{ij} for keyword j

- Pacing Eq.: $(\lambda_1, \dots, \lambda_n) \in [0,1]^n$ s.t.

- First price auction with bids $\lambda_i v_{ij}$

- For each agent i , if $\lambda_i < 1$ then total payment = B_i , else $\leq B_i$

- Equivalent to Fisher market with quasi-linear utilities!