



Security Games

CS580

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Commitment (Stackelberg strategies)

Commitment

Unique Nash equilibrium
(iterated strict dominance
solution)

commit →

U	NE 1, 1	3, 0
D	0, 0	2, 1
	L	R

B.R. ↓

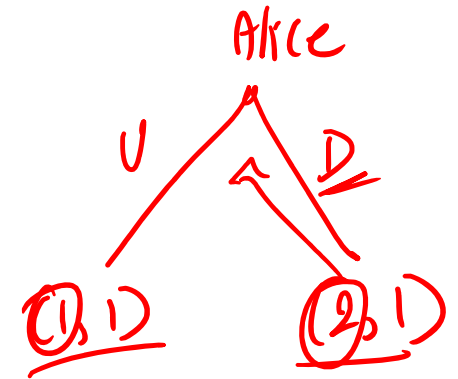
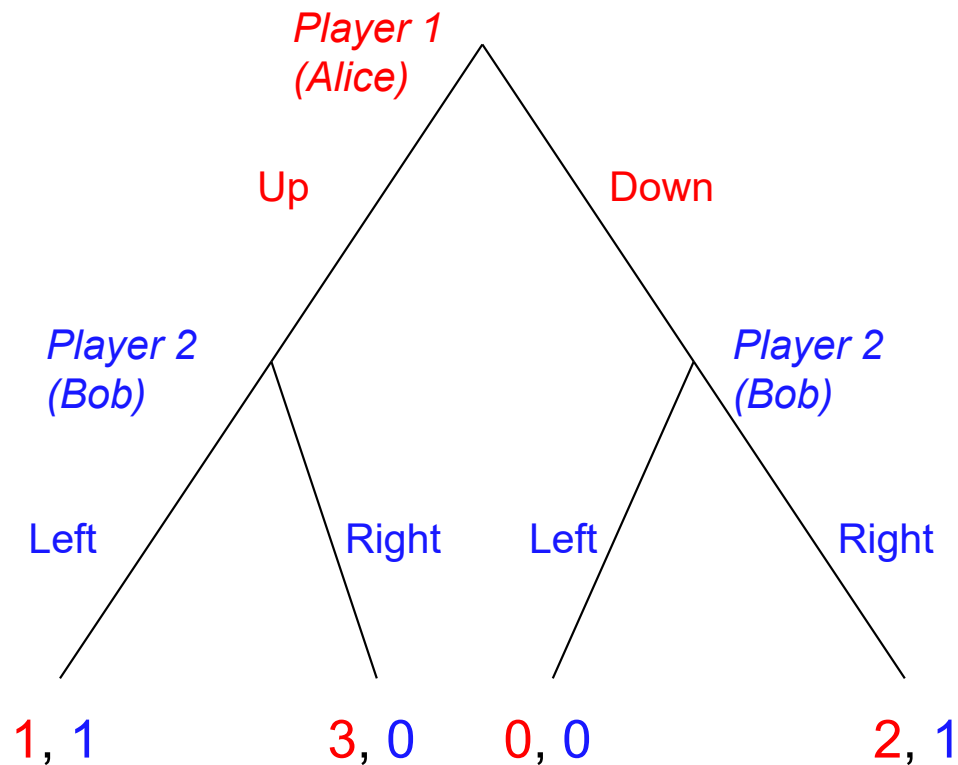


von Stackelberg

- Suppose the game is played as follows:
 - Alice commits to playing one of the rows,
 - Bob observes the commitment and then chooses a column
- Optimal strategy for Alice: commit to Down

Commitment: an extensive-form game

For the case of committing to a pure strategy:



Commitment to mixed strategies

	L	R
L	1, 1	3, 0
R	0, 0	2, 1

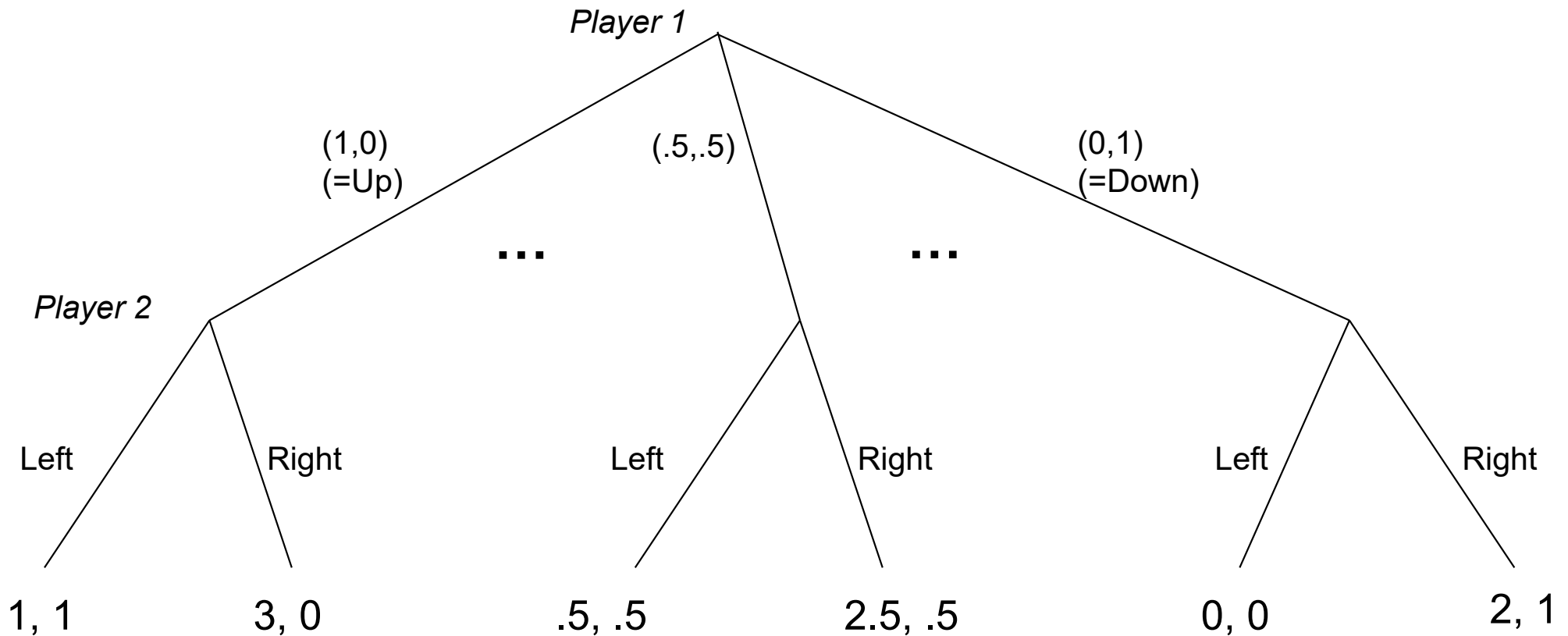
Handwritten red notes:

- 0.49 (above L column)
- 0.51 (above R column)
- 0.49 (to the left of L row)
- 0.51 (to the left of R row)
- Circle around (3, 0) and a line to 0.51

Also called a **Stackelberg (mixed) strategy**

Commitment: an extensive-form game

- ... for the case of committing to a mixed strategy:



- Economist: Just an extensive-form game, nothing new here
- Computer scientist: **Infinite-size game!** Representation matters

Computing the optimal mixed strategy to commit to

[Conitzer & Sandholm EC'06]

- Player 1 (Alice) is a leader.
- Separate LP for every column $j^* \in S_2$:

Alice plays $\bar{x}(j^*)$
where
 $j^* \in \arg\max_{j \in [n]} \text{OPT}(j)$

$\text{OPT}(j^*)$
and
 $\bar{x}(j^*)$ maximize $\sum_i x_i A_{ij^*}$ Alice's utility when Bob plays j^*
subject to $\forall j, (x^T B)_{j^*} \geq (x^T B)_j$ Playing j^* is best for Bob
 $x \geq 0, \sum_i x_i = 1$ x is a probability distribution

Among soln. of all the LPs,
pick the one that gives max utility.

On the game we saw before

$$\bar{x}(L) := (1, 0)$$

$$OPT(L) = 1$$

$$\text{maximize } 1x_1 + \cancel{0x_2}$$

subject to

$$1x_1 + \cancel{0x_2} \geq \cancel{0x_1} + 1x_2$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0, x_2 \geq 0$$

$OPT(L) < OPT(R)$
 $\therefore$ Play $\bar{x}(R) = (1/2, 1/2)$

	L	R
x_1	<u>1</u> , 1	3, 0
x_2	<u>0</u> , 0	2, 1

$$\bar{x}(R) = (1/2, 1/2)$$

$$OPT(R) = 2.5$$

$$\text{maximize } \underline{3}x_1 + 2x_2$$

subject to

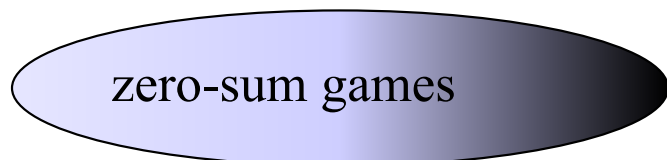
$$\cancel{0x_1} + 1x_2 \geq \underline{1}x_1 + \cancel{0x_2}$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0, x_2 \geq 0$$

Generalizing beyond zero-sum games

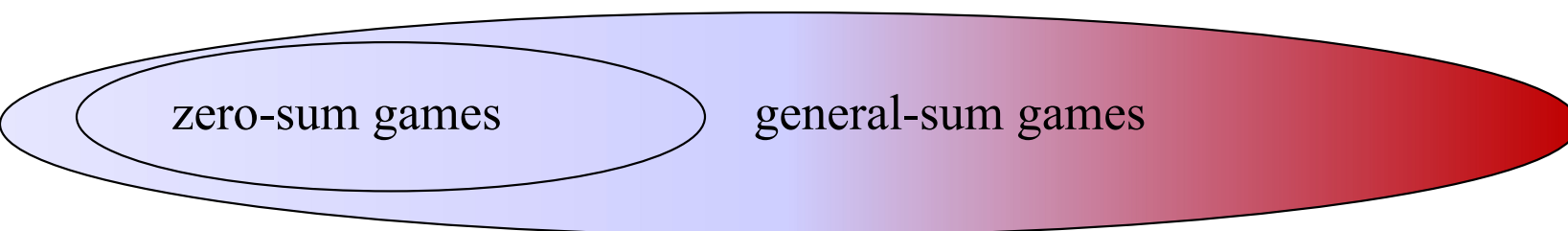
Minimax, Nash, Stackelberg all agree in zero-sum games



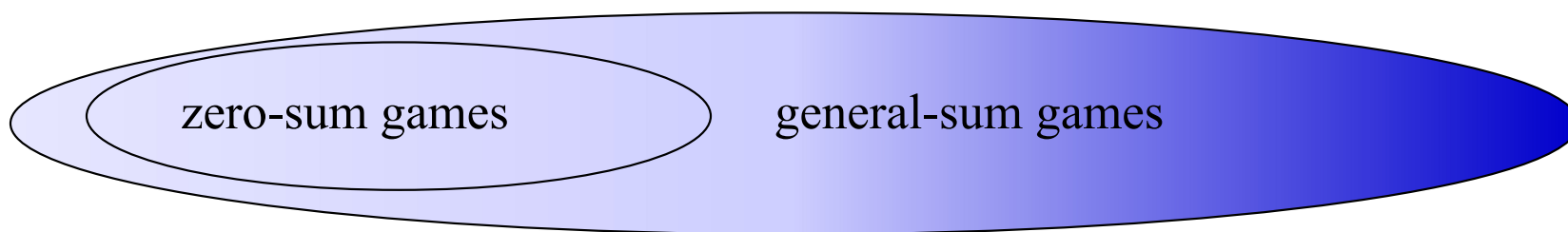
minimax strategies



0, 0	-1, 1
-1, 1	0, 0



Nash equilibrium



Stackelberg mixed strategies

Other nice properties of commitment to mixed strategies

- No **equilibrium selection** problem
- Leader's payoff **at least as good as** any Nash eq. or even correlated eq.
(von Stengel & Zamir [GEB '10])



0, 0	-1, 1
1, -1	-5, -5



$\geq$





Security Games (Stackelberg strategies)

See lecture scribbles,
and this paper.



Nash Bargaining

Nash Bargaining: Dividing Utilities

Two agents: 1, 2

Outside option utilities: c_1, c_2

Feasible set of utilities: $U \subseteq \mathbb{R}^2$ (convex),
 $(c_1, c_2) \in U$

Goal: define a bargaining function $f(c_1, c_2, U) \in U$
satisfying certain good properties

Nash Bargaining: Axioms

Two agents: 1, 2

Outside option with utilities: c_1, c_2

Feasible set of Utilities: $U \subseteq \mathbb{R}^2$ (convex), $(c_1, c_2) \in U$

Goal: $f(c_1, c_2, U) \in U$ that is

1. Scale free
2. Symmetric
3. Pareto Optimal
4. Independent of Irrelevant Alternatives (IIA)
5. Individually Rational

Nash Bargaining: Theorem

Two agents: 1, 2

Outside option with utilities: c_1, c_2

Feasible set of Utilities: $U \subseteq R^2$ (convex), $(c_1, c_2) \in U$

Goal: $f(c_1, c_2, U) \in U$ that is

1. Scale free
2. Symmetric
3. Pareto Optimal
4. Independent of Irrelevant Alternatives (IIA)
5. Individually Rational

Theorem (Nash'50). f satisfies the 5 axioms if and only if, $f(c_1, c_2, U)$ is

$$\begin{aligned} & \operatorname{argmax} (u_1 - c_1)(u_2 - c_2) \\ & \text{s.t.} \quad (u_1, u_2) \in U \\ & \quad u_1 \geq c_1, u_2 \geq c_2 \end{aligned}$$

Nash Bargaining: Theorem

Theorem (Nash'50). f satisfies the 5 axioms if and only if, $f(c_1, c_2, U)$ is

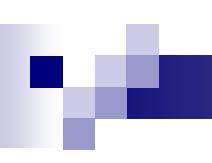
$$\begin{aligned} & \operatorname{argmax} (u_1 - c_1)(u_2 - c_2) \\ \text{s.t.} \quad & (u_1, u_2) \in U, u_1 \geq c_1, u_2 \geq c_2 \end{aligned}$$

Proof. ($\Leftarrow$)

1. Scale free
2. Symmetric
3. Pareto Optimal
4. Independent of Irrelevant Alternatives (IIA)
5. Individually Rational



Incomplete Information Bayesian Games



Bayesian Games

So far in Games,

- Complete information (each player has perfect information regarding the element of the game).

Bayesian Game

- A game with **incomplete information**
- Each player has initial **private information**, **type**.
- Bayesian equilibrium: solution of the Bayesian game

Bayesian game

- Utility of a player depends on her **type** and the actions taken in the game
 - θ_i is player i's type, $\theta_i \sim \Theta_i$. Utility when θ_i type and s play is $u_i(\theta_i, s)$
 - Each player knows/learns its own type, but only distribution of others (before choosing action)
 - Pure strategy $s_i: \Theta_i \rightarrow S_i$ (where S_i is i's set of actions)

(In general players can also receive signals about other players' utilities; we will not go into this)

		L	R
row player (Alice) type 1 (prob. 0.5)	U	4	6
	D	2	4

		L	R
row player type 2 (prob. 0.5)	U	2	4
	D	4	2

		L	R
column player (Bob) type 1 (prob. 0.5)	U	4	6
	D	4	6

		L	R
column player type 2 (prob. 0.5)	U	2	2
	D	4	2

Bayesian game

- Utility of a player depends on her **type** and the actions taken in the game
 - θ_i is player i 's type, $\theta_i \sim \Theta_i$. Utility when θ_i type and s play is $u_i(\theta_i, s)$
 - Knows only own type, but only distribution of others (before choosing action)
 - Pure strategy $s_i: \Theta_i \rightarrow S_i$ (where S_i is i 's set of actions)

(In general players can also receive signals about other players' utilities; we will not go into this)

Example: Single Item Auction

For player i

- Type: $v_i \sim D_i$
- Strategy: bid $b_i = s_i(v_i)$
- Utility $u_i(v_i, bids) = v_i - \text{payment}(bids)$

Bayes-Nash equilibrium

- A profile of strategies is a **Bayes-Nash equilibrium** iff

Mixed strategy of player i , $\sigma_i: \Theta_i \rightarrow \Delta(S_i)$

□ for every i , for every type θ_i , for every alternative action $z_i \in \Delta(S_i)$, we must have:

$$\sum_{\theta_{-i}} \underbrace{P(\theta_{-i})}_{\downarrow} u_i(\theta_i, \sigma_i(\theta_i), \sigma_{-i}(\theta_{-i})) \geq \sum_{\theta_{-i}} P(\theta_{-i}) u_i(\theta_i, z_i, \sigma_{-i}(\theta_{-i}))$$

$$\prod_{p \neq i} P(\theta_p)$$

Bayesian game

- Utility of a player depends on her **type** and the actions taken in the game
 - θ_i is player i's type, $\theta_i \sim \Theta_i$. Utility when θ_i type and s play is $u_i(\theta_i, s)$
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		L	R
row player (Alice) type 1 (prob. 0.5)	U	4	6
	D	2	4

		L	R
row player type 2 (prob. 0.5)	U	2	4
	D	4	2

		L	R
column player (Bob) type 1 (prob. 0.5)	U	4	6
	D	4	6

		L	R
column player type 2 (prob. 0.5)	U	2	2
	D	4	2

Converting Bayesian games to normal form

		L	R
row player	U	4	6
	D	2	4

type 1 (prob. 0.5)

		L	R
column player	U	4	6
	D	4	6

type 1 (prob. 0.5)

		L	R
row player	U	2	4
	D	4	2

type 2 (prob. 0.5)

		L	R
column player	U	2	2
	D	4	2

type 2 (prob. 0.5)

	type 1: L	type 1: L	type 1: R	type 1: R
	type 2: L	type 2: R	type 2: L	type 2: R
type 1: U	3, 3	3.5, 3	4, 4	5, 4
type 2: U				
type 1: U	4, 3.5	3, 3	4, 4.5	4, 4
type 2: D				
type 1: D	2, 3.5	3, 3	3, 4.5	4, 4
type 2: U				
type 1: D	3, 4	3, 3	3, 5	3, 4
type 2: D				

exponential
blowup in size

Car Selling Game

- A seller wants to sell a car
- A buyer has private value 'v' for the car w.p. $P(v)$
- Seller knows P , but not v
- Seller sets a price 'p', and buyer decides to buy or not buy.
- If sell happens then the seller gets p , and buyer gets $(v-p)$.

S_1 =All possible prices, $\Theta_1=\{1\}$

S_2 = {buy, not buy}, Θ_2 =All possible 'v'

$U_1(1, (p, \text{buy})) = p$, $U_1(1, (p, \text{not buy})) = 0$

$U_2(v, (p, \text{buy})) = v - p$, $U_2(v, (p, \text{not buy})) = 0$



Again what about corr. eq. in Bayesian
games?

Notion of signaling.

Look up the literature.