Final Exam Review

CS580

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Fair Division: EF and Relaxations

$[n]$: agents , M : indivisible items.

v_{ij} : Value of agent i for item j.

$$v_i(S) = \sum_{j \in S} v_{ij} \quad \text{(Additive)}$$

EF: Allocation $(A_1, \ldots, A_n)$

$$v_i(A_i) \geq v_i(A_k) \quad \forall k \in [n]$$

EF1: $v_i(A_i) \geq v_i(A_k \setminus j)$ $\quad \exists j \in A_k, \forall k \in [n]$

EFX: $v_i(A_i) \geq v_i(A_k \setminus j)$ $\quad \forall j \in A_k,$

PO:

☆ Round Robin.

$r = 1$

$A_1 : v_{11} \quad v_{1n+1}$

$\vdots$

$A_i : v_{i1}$

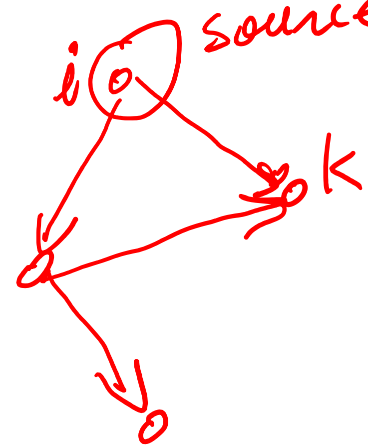
$\vdots$

$A_n : v_{in} \quad v_{i2n}$

☆ Envy-cycle Elimination.

Init: $A_i = \emptyset \quad \forall i$

Maintain Envy graph



if i envies k.

Next item to the source.

Fair Division: MMS

MMS value : $\max\limits_{(A_1, \ldots A_n) \text{ partition of } M} \ \min\limits_{k \in [n]} v_i(A_k)$

(μ_i)

$$\mu_i \leq \frac{v_i(M)}{n}$$

Fair Division: Problem

* chores allocation.

EF1. 9

d_{ij} : disutility of i for chore j

$d_i(S) = \sum_{j \in S} d_{ij}$

EF1: $(A_1, \ldots, A_n)$

$d_i(A_i \setminus c) \leq d_i(A_k)$

$\exists c \in A_i$

	r_1	r_2	r_3	r_k
A_1	d_{i1}	$d_{i(n+1)}$	$\ldots$	$d_{i((k+1)n+1)}$ ✗
.	d_{i2}	$\vdots$		
A_i	$\vdots$	$\vdots$		
A_n	d_{in}	d_{i2n}		

$d_{i((k-1)n+i)}$

$d_{i1} \leq d_{1i} \quad \forall i \in [n]$

~~Two-player~~ Games

$[n]$: players. S_i : strategy set of player i

$s_i \in S_i$: $s = (s_1, \dots, s_n)$ is a NE.

$\forall i \in [n]$ $\quad U_i(s_i, s_{-i}) \geq U_i(s_i', s_{-i})$ $\quad \forall s_i' \in S_i$

(weak)
(strong). Dominant strategy

Weak: $\quad U_i(s_i, s_{-i}) \geq U_i(t, s_{-i}) \quad \forall s_{-i} \quad \forall t \in S_i$

Strong: $\quad >$

Stackleberg Games

A Leader : plays x $\{1, \dots, m\}$

B Follower: y = Best response to x. $\{1, \dots, n\}$

$j \in [n]$: LP(j) : max: $(x^T A)_j$

$(x^T B)_j \geq (x^T B)_k \quad \forall k \in [n]$

$\max_{j \in [n]}$ LP(j)

$\sum_i x_i = 1$

$x_i \geq 0$

Fixing x, $\underbrace{\text{find } y(x)}_{\Downarrow}$

$\arg\max_x$: Leader's payoff $(x, y(x))$

Games: Problem

$$SW(S) = \ell_{i^*}(S) \le \ell_k(S) + p_j \quad \forall k \in M$$

$$j : s_j = i^*$$

$$\Downarrow$$

$$\sum_{k \in M} \ell_{i^*}(S) \le \sum_{k \in M} \ell_k(S) + \sum_{k \in M} p_j$$

$$\cancel{m} \cdot \ell_{i^*}(S) \le \frac{\sum_{j \in [n]} p_j}{m} + \cancel{m}\, p_j$$

$$\le OPT \qquad \le OPT$$

$$\Rightarrow \quad SW(S) = \ell_{i^*}(S) \le 2\,OPT$$

$$PoA = \frac{\ell_{i^*}(S)}{OPT} \le 2$$

Routing Games: PoA

Load Balancing: n jobs, set M of m machines

$j \in [n]$: $S_j = M$; p_j: processing time.

$s = (s_1, \dots, s_n)$

$\ell_i(s) = \sum_{j \in [n],\, s_j = i} p_j$ $\quad D_j(s) = \ell_{s_j}(s)$

$SW(s) = \max_{i \in M} \ell_i(s)$ $\quad$ OPT.

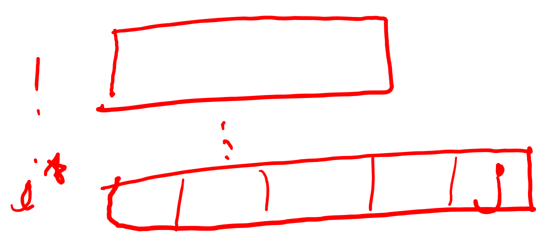
PoA ≤ 2 ?

claim 1: OPT $\ge \sum_j p_j / m$, OPT $\ge \max_{j \in [n]} p_j$

NE: $s = (s_1, \dots, s_n)$ $\quad s \quad i^* = \arg\max_{i \in M} \ell_i(s)$ $\quad \forall k \in M$

$D_j(s_j, s_{-j}) \le D_j(k, s_{-j})$ $\forall k \in M$

$\ell_k(s) + p_j \ge \ell_{i^*}(s)$



Atomic Routing Games: Potential

Recall Potential : ϕ

PoS : $s^* = \arg\min_s \phi(s)$

$C(s^*) \le \phi(s^*) \le \phi(s^{OPT}) \le \alpha\, C(s^{OPT})$

PoA $\dfrac{\text{cost}(s^{NE\text{-}best})}{OPT} \le \dfrac{\text{cost}(s^*)}{OPT} \le \alpha$

Cost-sharing Games: PoS

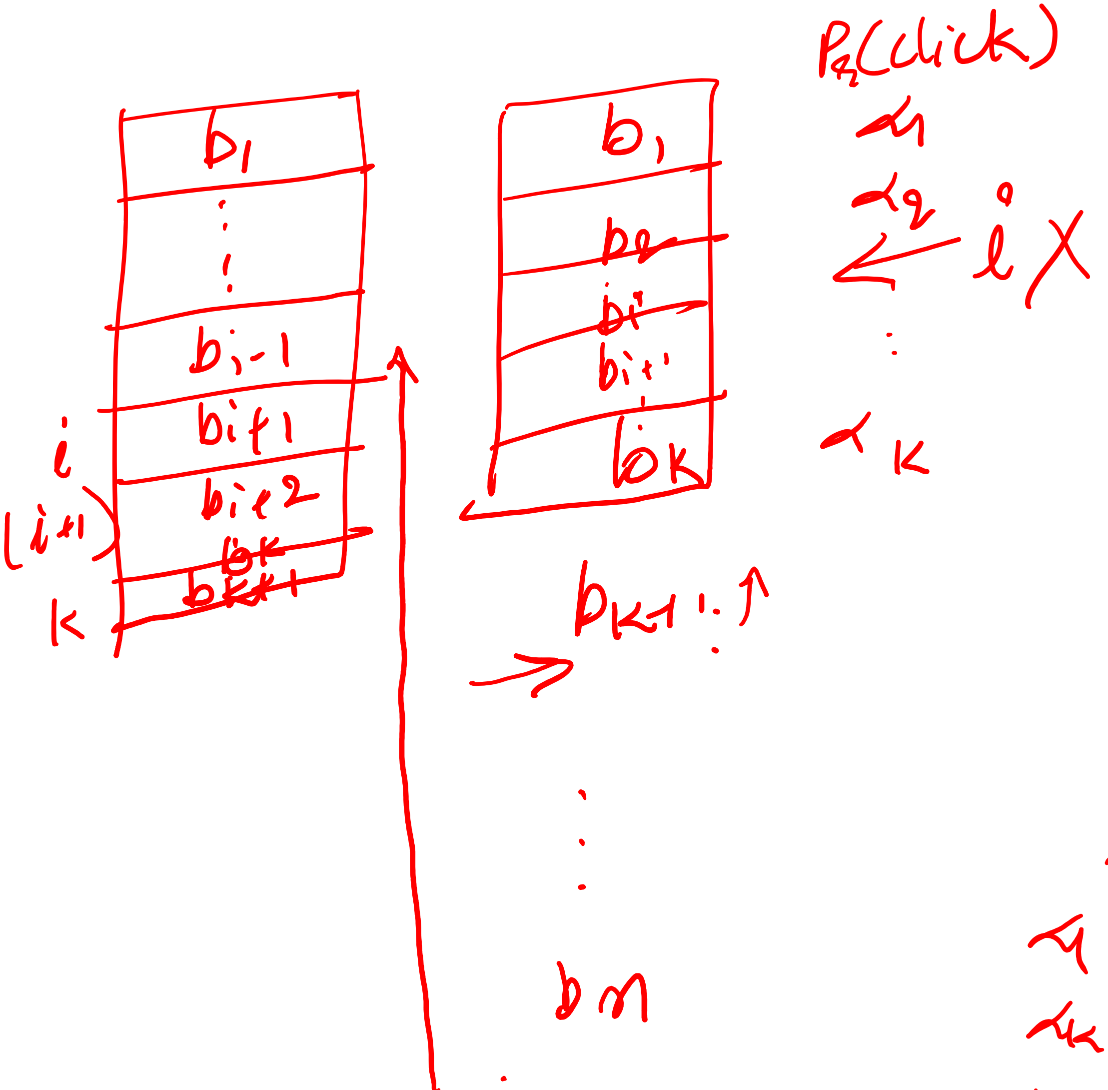
Problem

Single Item Auction

Second Price: DSIC.

bidding $b_i = v_i$ is dominant strategy $\forall i$.

Keyword (pay-per-click) Auction: GSP

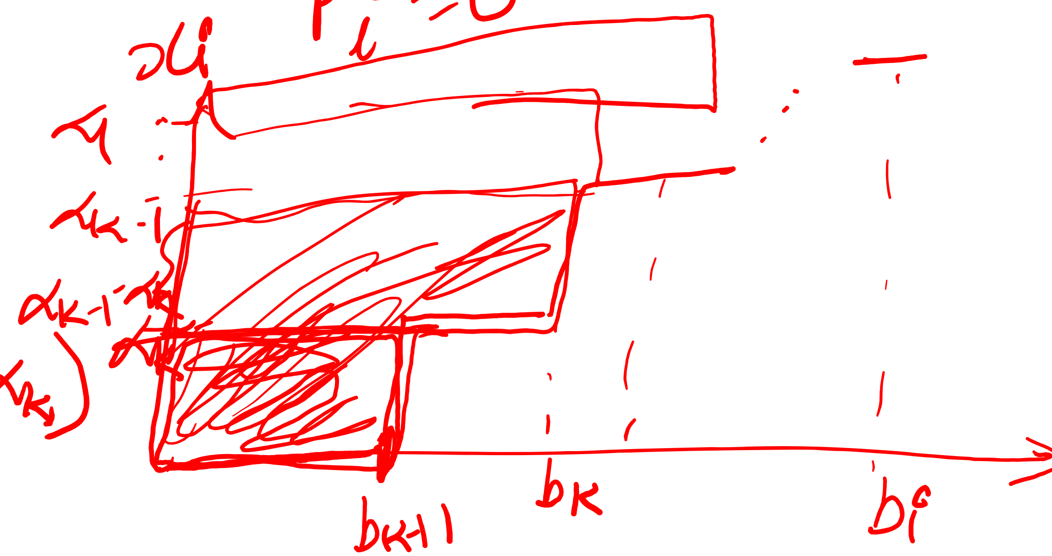


$\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_K$

$b_1 \ge b_2 \ge \dots \ge b_K$

$p_i^{GSP} = b_{i+1} \quad \forall i \le K$

$p_i = 0 \quad \forall i > K$



$p_i^{DSIC} = b_{K+1}\cdot\alpha_K + b_K(\alpha_{K-1}-\alpha_K)$

Single Parameter Auction

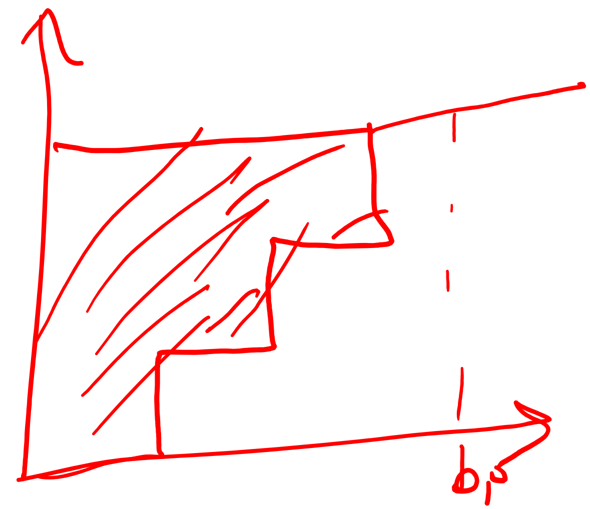
$(b_1, \ldots, b_n) \rightarrow$ Mechanism $\rightarrow (x_1(b), \ldots, x_n(b))$

$\rightarrow (p_1(b), \ldots, p_n(b))$

Fix i, b_{-i}:

DSE iff $x_i(b_i, b_{-i})$ is monotone in b_i

iff $p_i(b_i, b_{-i})$



Problem

Myerson's Max.Revenue Auction : Single Parameter

$V_i \sim D_i$

f_i $\quad F_i$

$$\mathrm{Rev} = \text{Virtual-SW}.$$

$$= \max_{x \in X} \sum_i^n \underline{\phi_i(b_i)\, x_i}$$

$$\phi_i(v_i) = v_i - \frac{1 - F_i(v_i)}{f_i(v_i)}$$

b_i : bid

bid b_i: $\phi_i(b_i) \geq 0$

$$\Rightarrow \quad b_i \geq \phi_i^{-1}(0) = \text{Reserve Price against } i$$

Myerson’s Max. Revenue Auction

VCG Auction : Recall.

Θ : set of outcomes

$V_i : \Theta \to \mathbb{R}_+$:

① S.W. max outcome:

$$\omega^* \in \arg\max_{\omega \in \Theta} \sum_{i \in [n]} V_i(\omega)$$

② Payment P_i = Harm caused by i's participation to others

$$= \max_{\omega \in \Theta} \sum_{k \neq i} V_k(\omega) - \sum_{k \neq i} V_k(\omega^*)$$

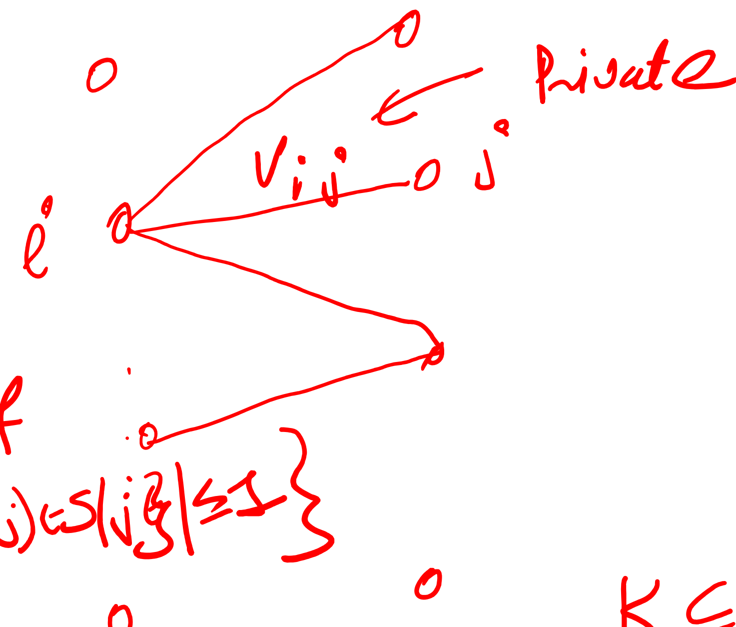
Problem

k: # edges that can be picked.

b_{ij}: bid of agent i for connecting to target j

Agents E Targets

at most one agent at least



Θ : $\{S \subseteq E \mid |S| = k \;\&\; |\{(i,j) \in S \mid j\}| \leq 1\}$

$K \subseteq E$

$v_i(S) = \max_{(i,j) \in S} v_{ij}$

$S^* \in \arg\max_{S \in \Theta} \sum_{i \in [n]} v_i(S)$

Obs 1: S^* will have at most one edge for each i