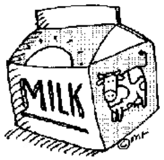
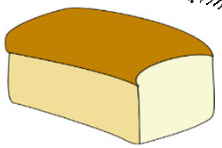
CS 580: Topics on AGT

Lec 2: Fair Division of Divisibles via Competitive Equilibirum

Instructor: Ruta Mehta

Divisible goods





Goal: Find *fair* and *efficient* allocation

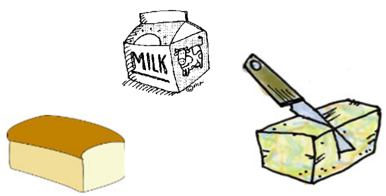
n.


ASSETS
LIABILITIES


U.S.ARMY
UNITED STATES COAST GUARD
1790
U.S.AIR FORCE
MARINE CORPS
NAVY

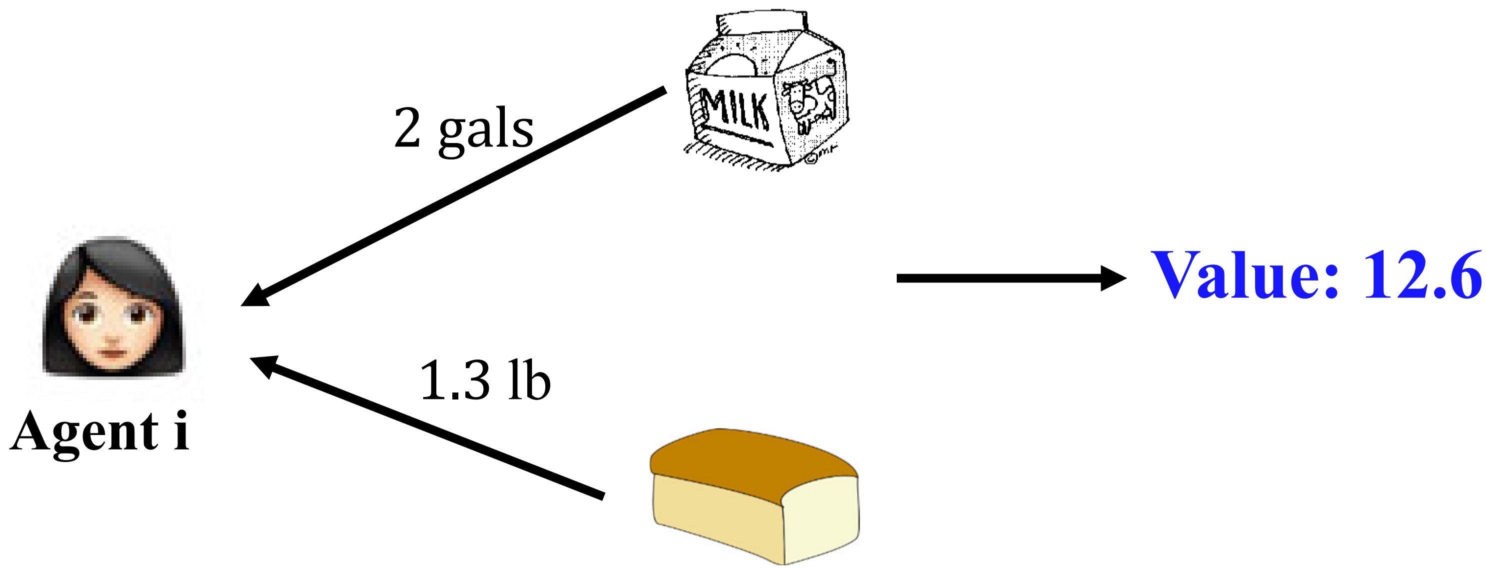
Model

- A: set of n agents
- M: set of m divisible goods (manna)



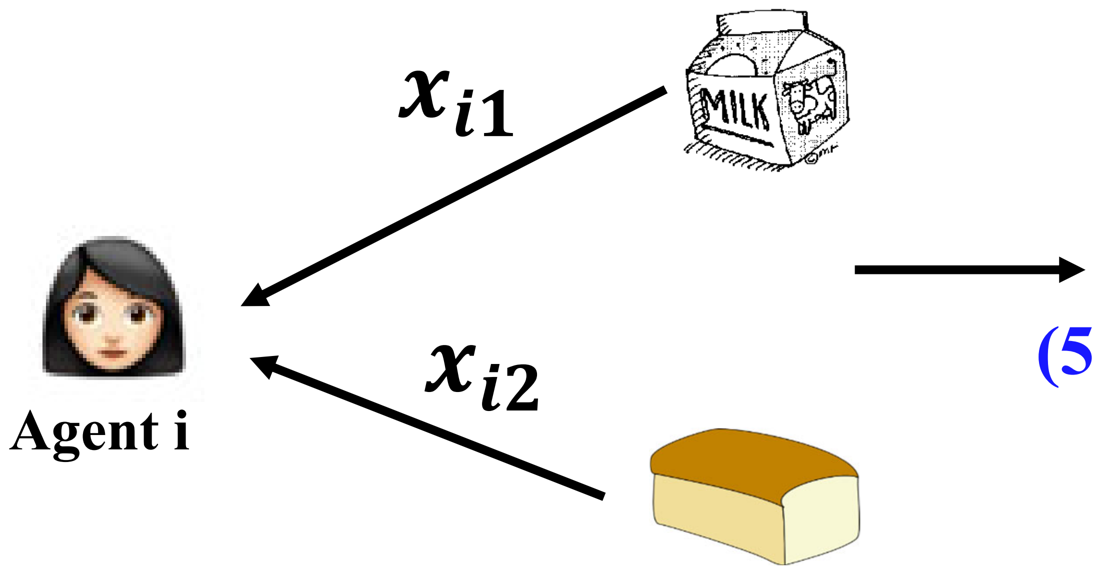
- Each agent i has
  - Valuation function $V_i: R_+^m \rightarrow R_+$ over bundles of items

Valuation function



Values milk at 5/gallon, and bread at 2/lb

Valuation function



Value:
$(5 * x_{i1}) + (2 * x_{i2})$

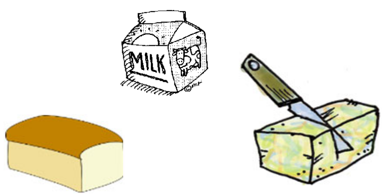
Values milk at 5/gallon, and bread at 2/lb

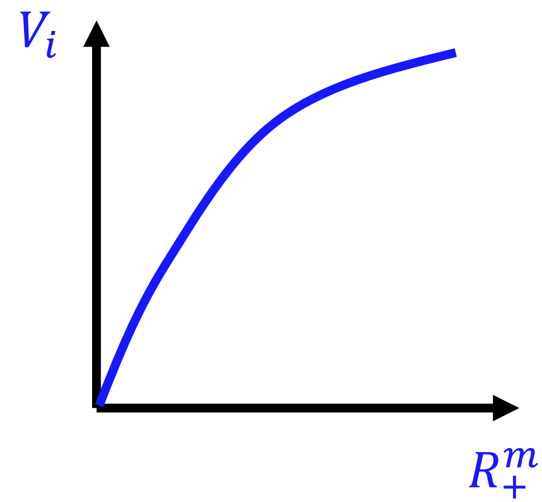
Linear/Additive Valuation

Model

- A: set of n agents

- M: set of m divisible goods (manna)

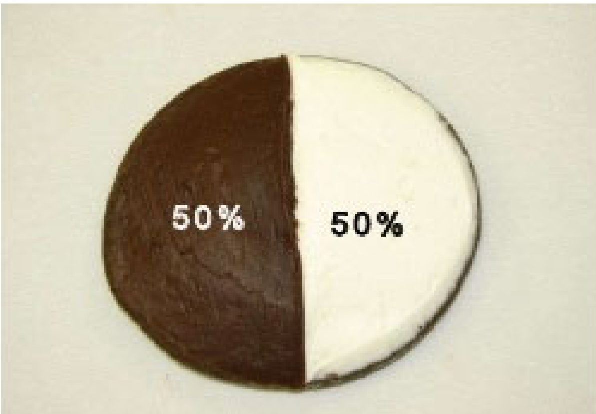




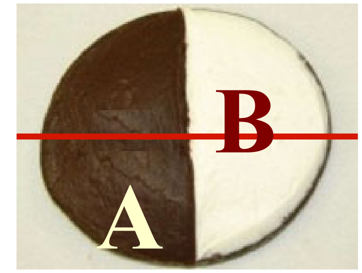
- Each agent i has
  - Valuation function $V_i: R_+^m \to R_+$ over bundles of items
  - Concave: Captures ***decreasing marginal returns***.

Goal: Find *fair* and *efficient* allocation

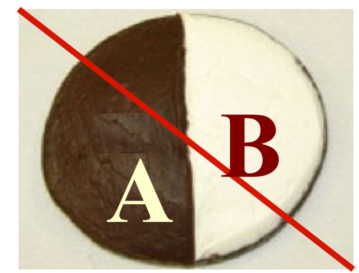
Example: Half moon cookie

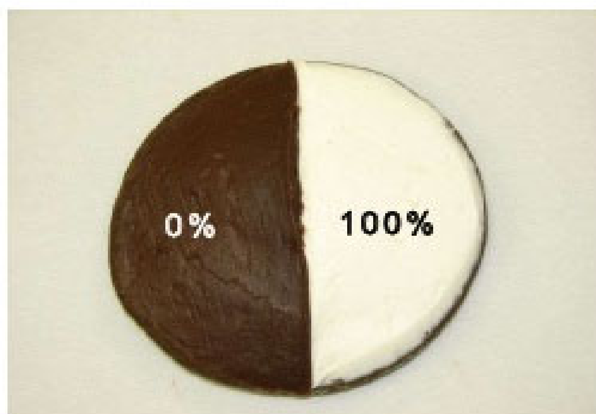


(i)

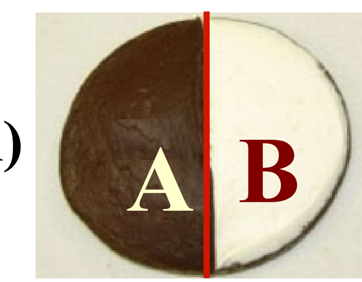


(ii)





(iii)



Fair (Agreeable)

Efficient (Non-wasteful)

Allocation: Bundle $X_i \in R_+^m$ to agent i

Envy-free: No agent *envies* other's allocation over her own.

For each agent i,
$V_i(X_i) \geq V_i(X_j), \forall j \in [n]$

Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

For each agent $i, V_i(X_i) \geq \frac{V_i(M)}{n}$

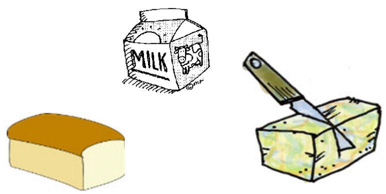
Pareto-optimal: No other allocation is better for all.

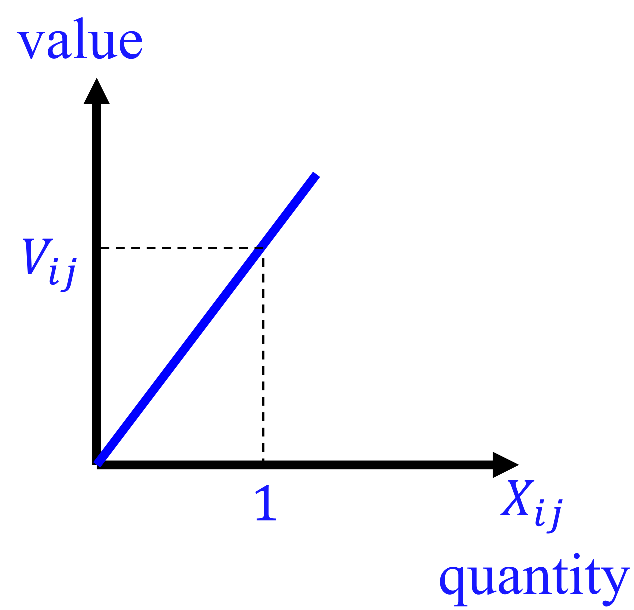
There is no Y, s. t.
$V_i(Y_i) \geq V_i(X_i), \forall i \in [n]$

Welfare Maximizing
$(max: \sum_i V_i)$

Model

- A: set of n agents
- M: set of m divisible goods (manna)





- Each agent i has
  - **Additive/linear $V_i: R_+^m \to R_+$:**

$$V_i(X_{i1}, \ldots, X_{im}) = \sum_{j \in M} V_{ij} X_{ij}$$

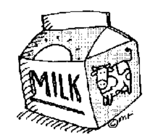
Fair (Agreeable)

Efficient (Non-wasteful)

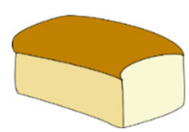
Envy-free: **No agent *envies* other's allocation over her own.**

Proportional: **Each agent i gets value at least $\frac{V_i(M)}{n}$**

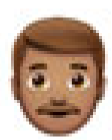
[3, 2, 2]
[0, 0, 0]



Allocation in red



[20, 20, 30]
[0, 0, 0]



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Proportional: Each agent i gets value at least $\frac{V_i(M)}{n}$

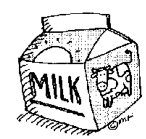
Efficient (Non-wasteful)

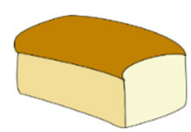
Pareto-optimal: No other allocation is better for all.

[3, 2, 2]
[1/2, 1/2, 1/2]

Allocation in red

[20, 20, 30]
[1/2, 1/2, 1/2]





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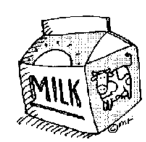
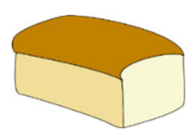
Welfare Maximizing $(max: \sum_i V_i)$

[3, 2, 2]
[1, 1/2, 0]



Allocation in red



[20, 20, 30]
[0, 1/2, 1]

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Welfare Maximizing $(max: \sum_i V_i)$

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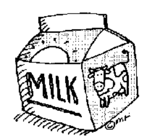
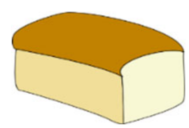
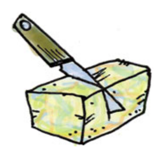
[0, 0, 0]





Allocation in red

[20, 20, 30]
[1, 1, 1]



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Efficient (Non-wasteful)

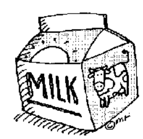
Pareto-optimal: No other allocation is better for all.

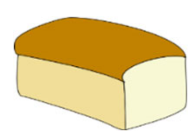
(Nash) Welfare Maximizing ($\Pi_i V_i$)

Allocation in red

[3, 2, 2]
[1, 1/2, 0]

[20, 20, 30]
[0, 1/2, 1]

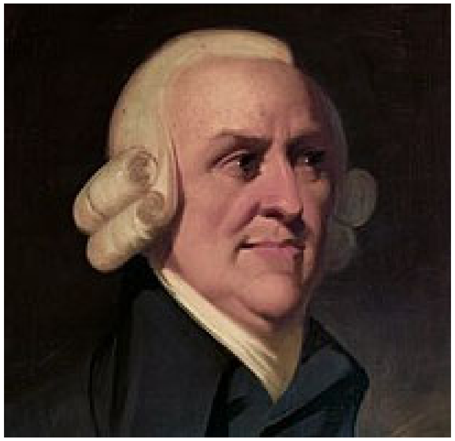




Fair (Agreeable)	Efficient (Non-wasteful)
Envy-free	Pareto-optimal
Proportional	(Nash) Welfare Maximizing

Competitive Equilibrium
(with equal income)

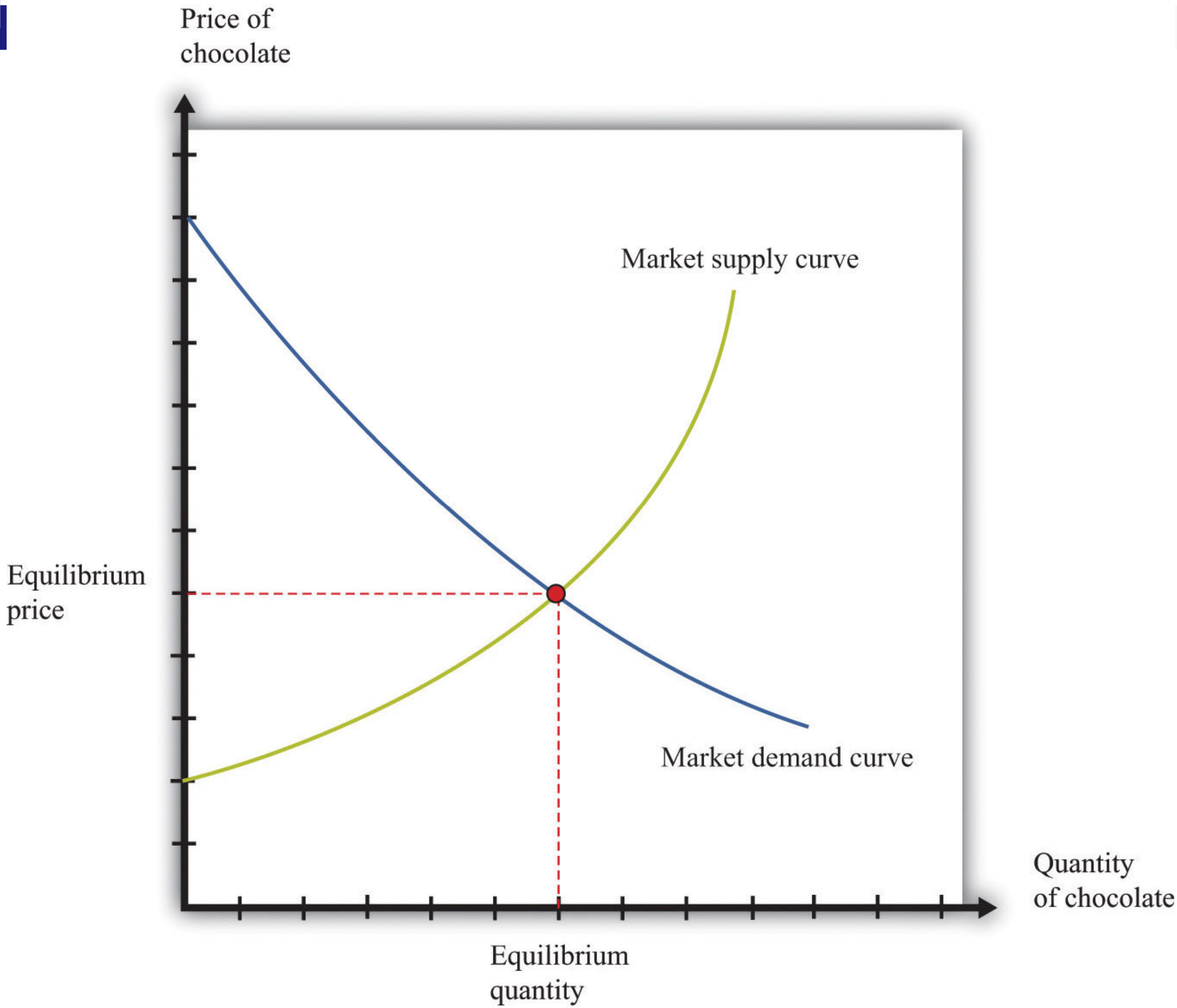
Beginning of Competitive Equilibrium



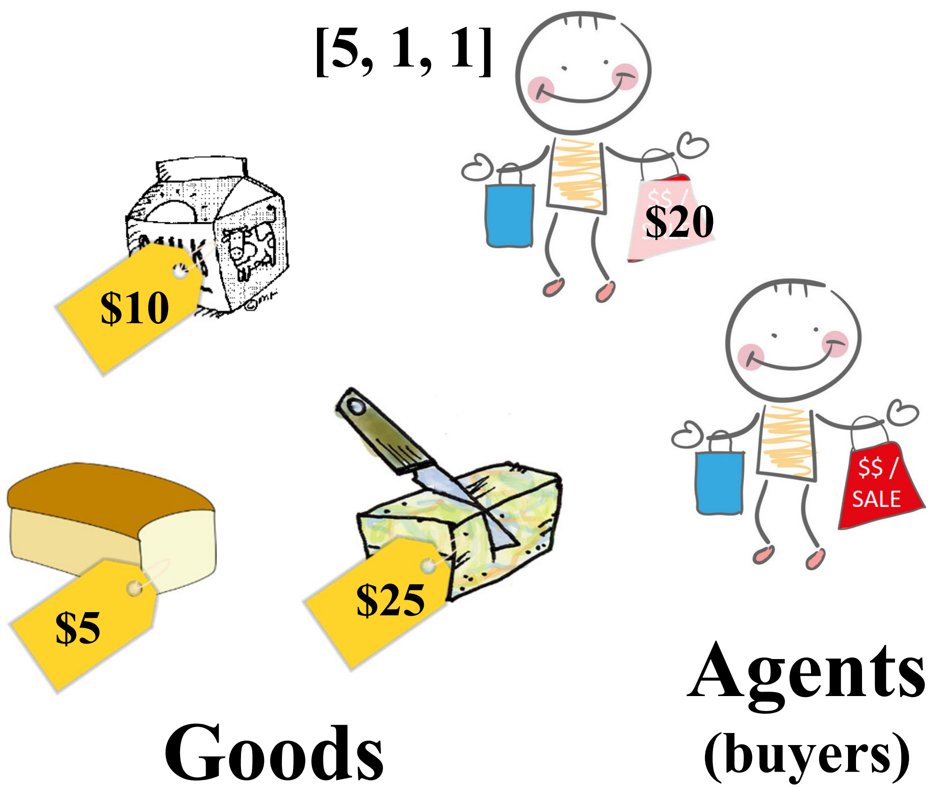
Adam Smith (1776)

Invisible hand

"Economic concept that describes the unintended greater social benefits and public good brought about by individuals acting in their own self-interests.[1][2] The concept was first introduced by Adam Smith in *The Theory of Moral Sentiments*, written in 1759. According to Smith, it is literally divine providence, that is the hand of God, that works to make this happen."


Price of chocolate
Market supply curve
Equilibrium price
Market demand curve
Quantity of chocolate
Equilibrium quantity

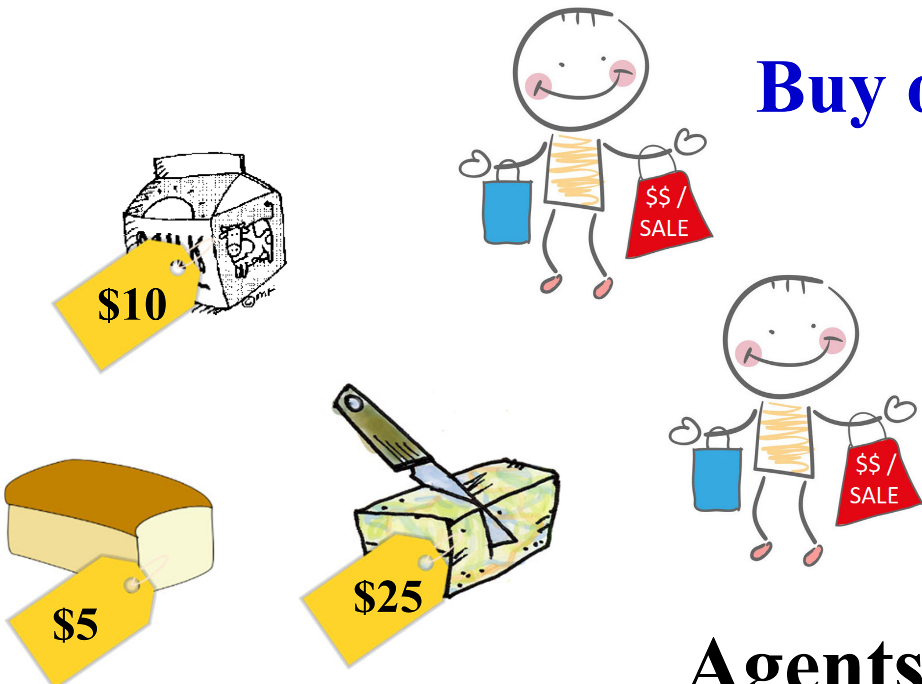
Competitive (market) Equilibrium (CE)



Demand optimal bundle

$argmax_{\{X \text{ affordable}\}} V_i(X)$

Competitive (market) Equilibrium (CE)



Goods

Agents
(buyers)

Buy optimal bundle ⟶ **Demand**

Competitive Equilibrium:
Demand = Supply

CE Example

Demands

[2, 0]

Total Demand

Supply

[5, 1]

1

2 > 1

Demand > Supply

[1, 4]

1

1

[0, 1]

CE Example

[1, 0]

Demand

[5, 1]

1

Demand = Supply
⇒ CE

[1, 4]

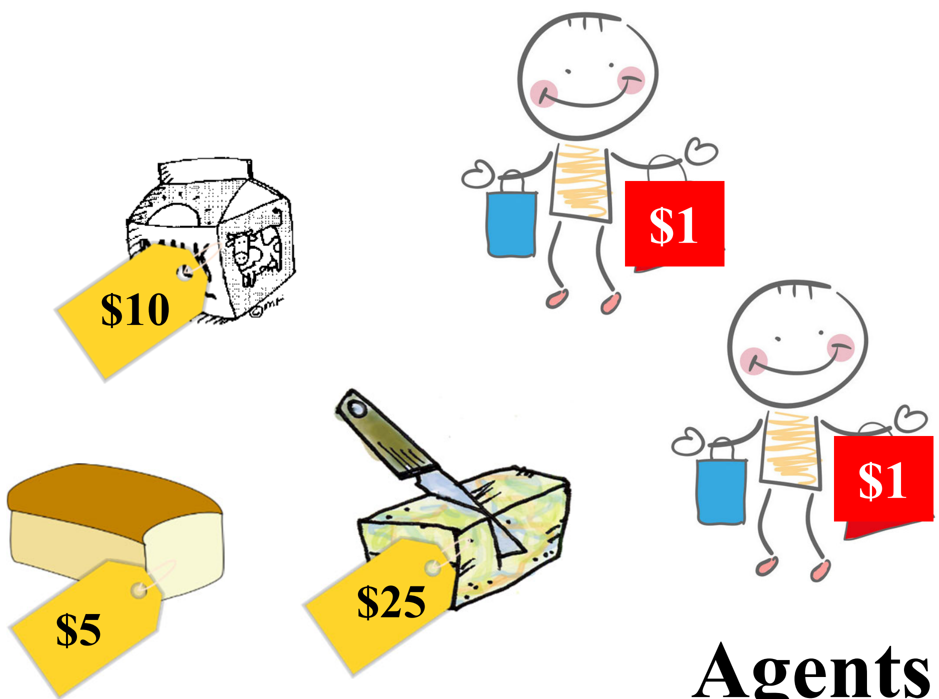
1

[0, 1]

w/ equal income (CEEI):

Agents have the same amount of money

CEEI: Properties



Goods

Agents
(buyers)

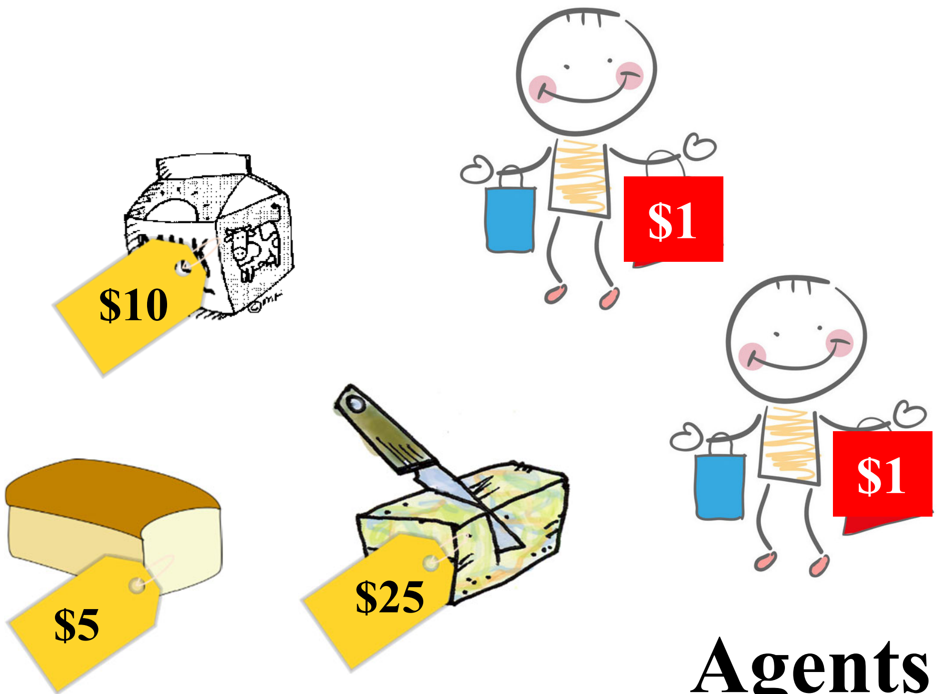
Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

An agent can afford anyone else's bundle, but demands her own
⇒ Envy-free

1^{st} **welfare theorem**
⇒ Pareto-optimal

CEEI: Properties



Goods

Agents (buyers)

Demand optimal bundle

Competitive Equilibrium:
Demand = Supply

Envy-free & "Demand=Supply"
⇒ Proportional

Proof.

Envyfree

$\Rightarrow V_i(\bar{X}_i) \geq V_i(\bar{X}_k), \forall k \in [n]$

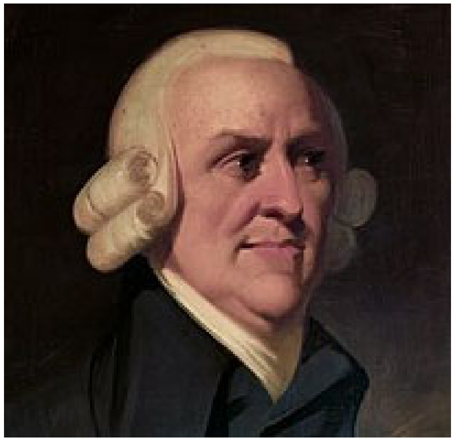
$\Rightarrow nV_i(\bar{X}_i) \geq \sum_{k\in[n]} V_i(\bar{X}_k)$

"Demand = Supply"

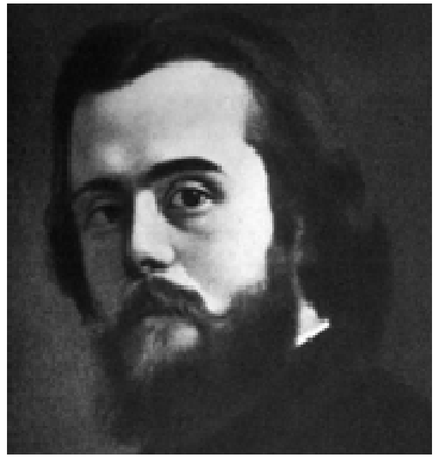
$$\Rightarrow \sum_{k\in[n]} V_i(\bar{X}_k) \geq V_i(M) \ (\because V_i \text{ concave})$$

$$\Rightarrow V_i(\bar{X}_i) \geq \frac{V_i(M)}{n}$$

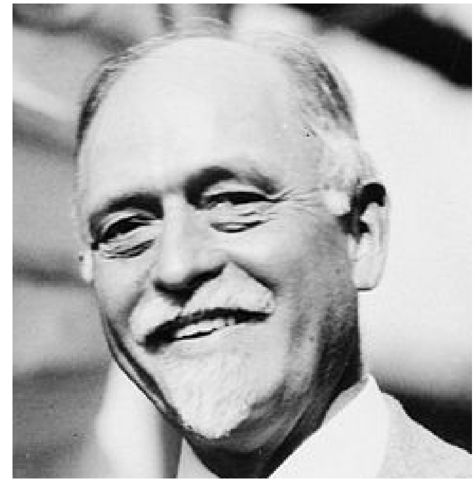
CE History



Adam Smith
(1776)



Leon Walras
(1880s)



Irving Fisher (1891)

Arrow-Debreu (1954)

(Nobel prize)

(Existence of CE in the exchange model w/ firms)

…

Computation of CE (w/ goods)

Algorithms

- Convex programming formulations
  - Eisenberg-Gale (1959): CEEI w/ 1-homogeneous valuations
  - Shmyrev (2009), DGV (2013), CDGJMVY (2017) …
- (Strongly) Poly-time algorithms (linear valuations)
  - DPSV (2002), Orlin (2010), DM (2015), GV (2019) …
- Simplex-like algorithms: Eaves (1976), GM.SV (2011), GM.V (2014), …

Complexity

- PPAD: Papadimitrou'92, CDDT'09, VY'11, CPY'17, Rubinstein'18, …
- FIXP: EY'09, GM.VY'17, F-RHHH'21 …

Learning: RZ'12, BDM.UV'14, …, FPR'22, …

Matching/mechanisms: BLNPL'14, …, KKT'15, …, FGL'16, …, AJT'17, …, BGH'19, BNT-C'19, …

***Alaei, Bei, Branzei, Chen, Cole, Daskalakis, Deng, Devanur, Duan, Dai, Etessami, Feldman, Fiat, Filos-Ratsikas, Garg, Gkatzelis, Hansen, Hogh, Hollender, Jain, Jalaly, Hoefer, Kleinberg, Lucier, Mai, Mehlhorn, Mehta, Mansour, Morgenstern, Nisan, Paes, Lee, Leme, Papadimitriou, Paparas, Parkes, Roth, Saberi, Sohoni, Talgam-Cohen, Tardos, Vazirani, Vegh, Yazdanbod, Yannakakis, Zhang,… … …**

Simple Tatonnement Procedure (Algo)

Increase prices of the over demanded goods.

Theorem. Tatonnement process Converges to a CE if $V_i s$ are *weak gross substitutes (WGS)*.

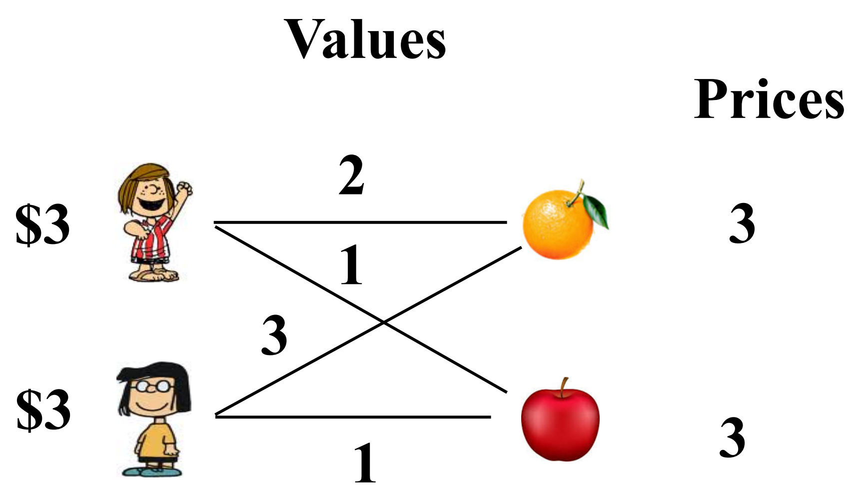
WGS: Increase in price of a good does not decrease demand of any other good.

Example: Linear $V_i s$

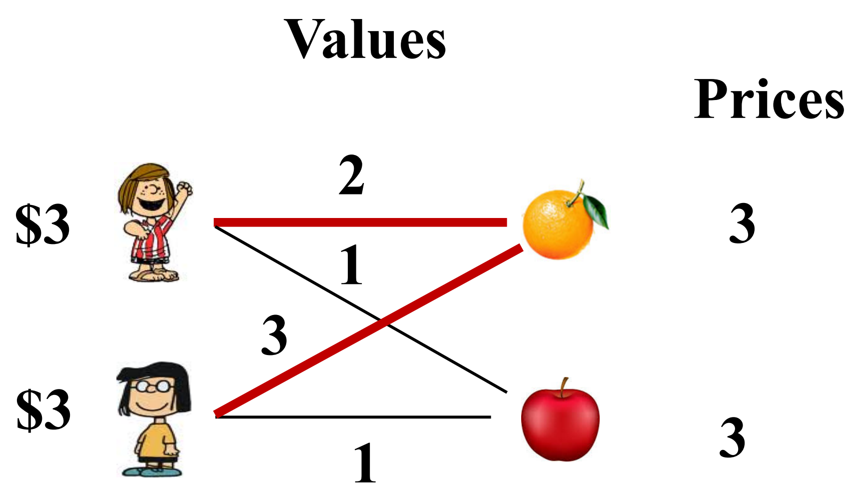
$$V_i(X_i) = \sum_{j \in [m]} V_{ij} X_{ij}$$

Fast Computation: Characterization

Example (Intuition)

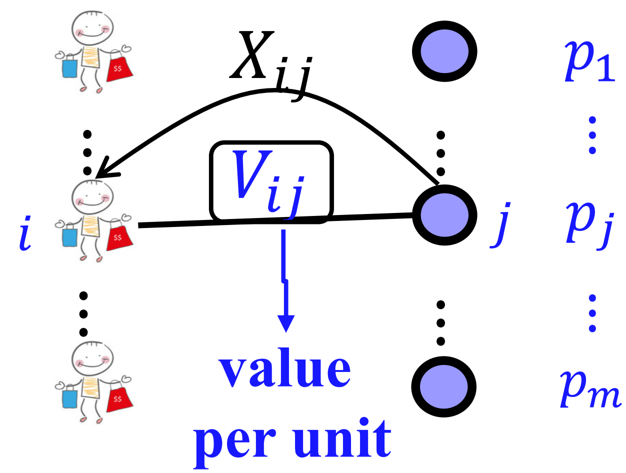


Example (Intuition)



Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



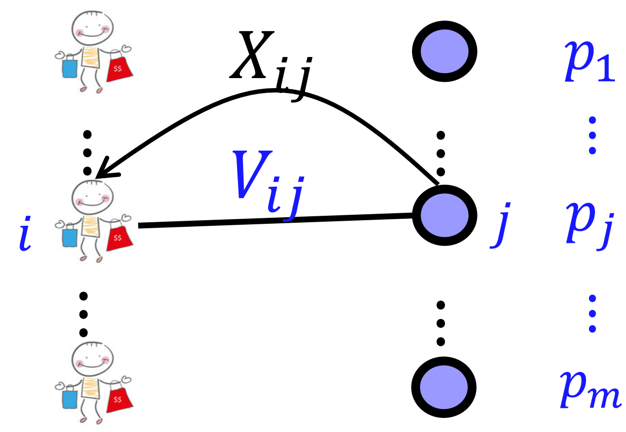
Optimal bundle: can spend at most *one* dollar.

Intuitition

spend wisely: on goods that gives maximum value-per-dollar $\frac{V_{ij}}{p_j}$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollar.

$$\sum_{j \in M} V_{ij} X_{ij} = \sum_j \frac{V_{ij}}{p_j} (p_j X_{ij}) \le \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) \sum_j p_j X_{ij} \le \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) 1$$

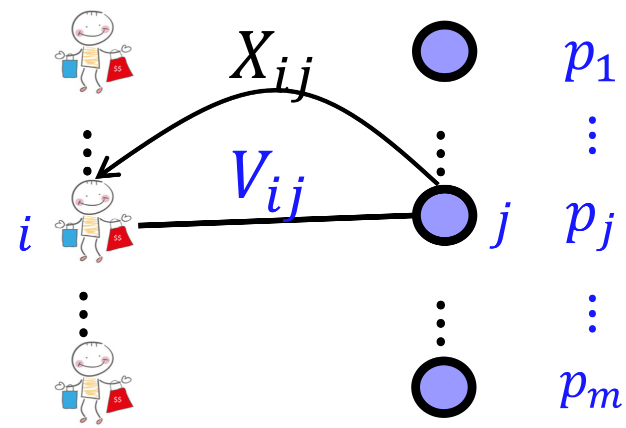
($ spent)

value per dollar spent
(bang-per-buck)

MBB
Maximum
bang-per-buck

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollar.

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value per dollar spent
(bang-per-buck)

= iff

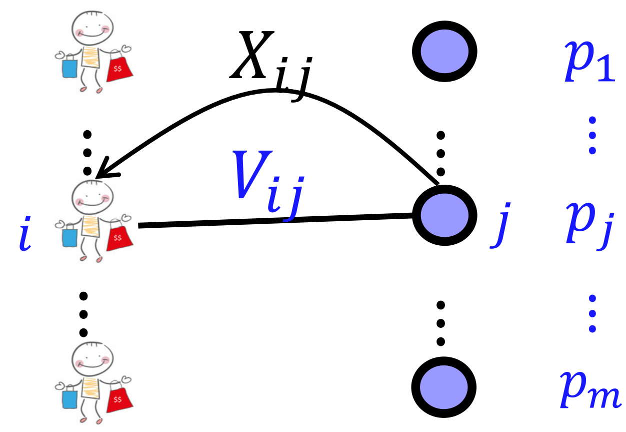
MBB
Maximum
bang-per-buck

Buy only MBB goods.

$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



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$$\sum_{j \in M} V_{ij} X_{ij} = \sum_j \frac{V_{ij}}{p_j}(p_j X_{ij}) \le \left(\max_{k \in G} \frac{V_{ik}}{p_k}\right) \sum_j p_j X_{ij} \le \left(\max_{k \in G} \frac{V_{ik}}{p_k}\right) 1$$

value per dollar spent
(bang-per-buck)

= iff

= iff

MBB
Maximum
bang-per-buck

Buy only MBB goods.

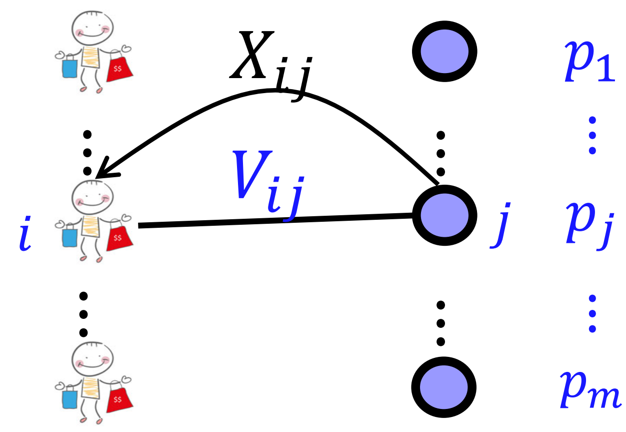
$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

Spends all of 1 dollar.

$$\sum_j p_j X_{ij} = 1$$

Linear Valuations: CEEI

$$V_i(X_i) = \sum_{j \in M} V_{ij} X_{ij}$$



Optimal bundle: can spend at most *one* dollars.

$$\sum_{j \in M} V_{ij} X_{ij} \leq \left(\max_{k \in G} \frac{V_{ik}}{p_k} \right) 1$$

$=$

iff

1. Buy only MBB goods.

$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = MBB$$

2. Spends all of 1 dollar.

$$\sum_j p_j X_{ij} = 1$$

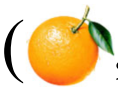
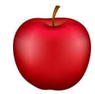
Linear $V_i s$: CEEI Characterization

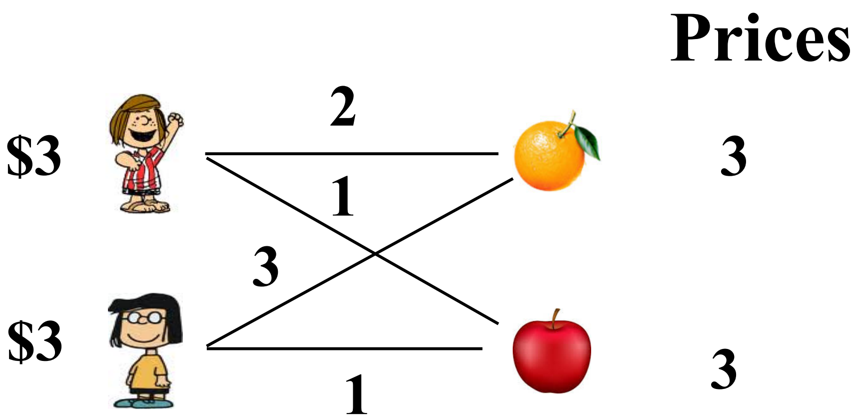
Pirces $p = (p_1, \dots, p_m)$ and allocation $X = (\bar{X}_1, \dots, \bar{X}_n)$ are at equilibrium iff

- Optimal bundle (OB): For each agent i
  - $\sum_j p_j X_{ij} = 1$
  - $X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = \max_{k \in M} \frac{V_{ik}}{p_k}$, for all good j
- Market clears: For each good j,

$$\sum_i X_{ij} = 1.$$

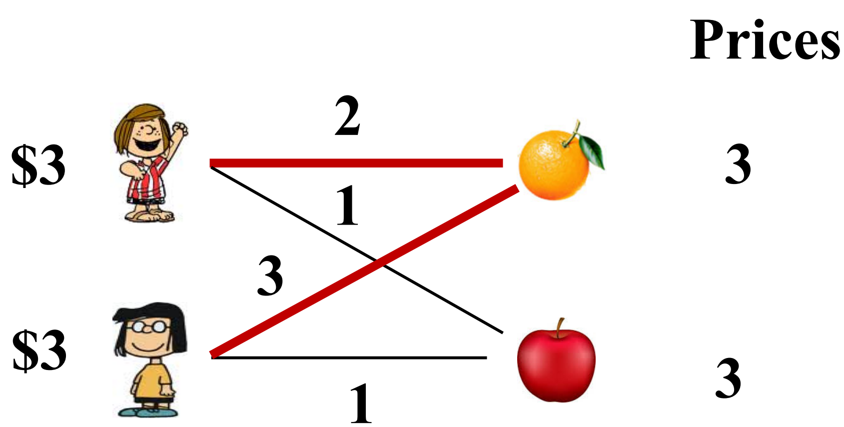
Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $3 and a linear utility function



Example

- 2 Buyers (,), 2 Items (,) with unit supply
- Each buyer has budget of \$3 and a linear utility function

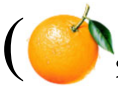
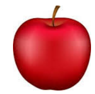


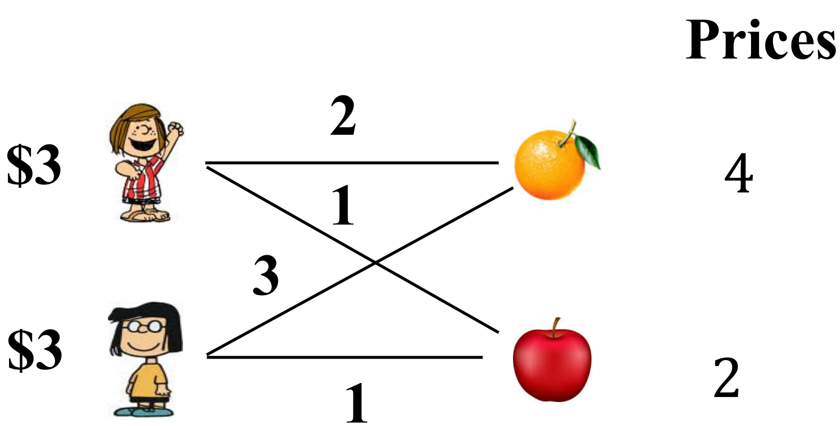
Demand ≠ Supply

MBB

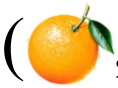
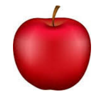
Not an Equilibrium!

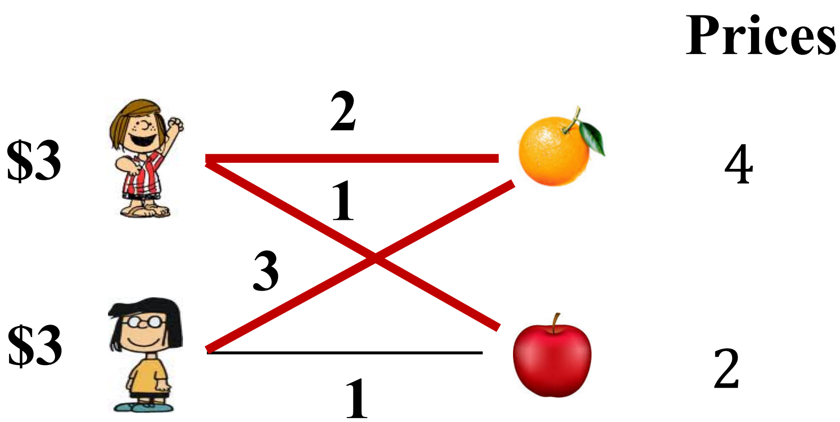
Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $3 and a linear utility function



Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $3 and a linear utility function



Demand = Supply

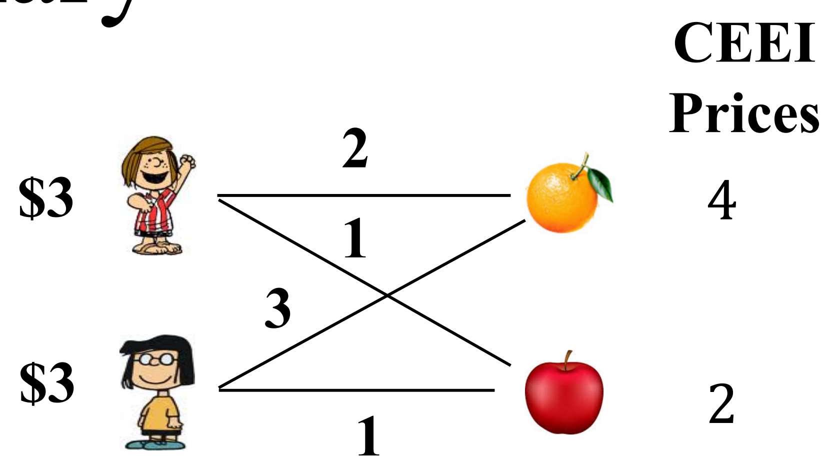
MBB

Equilibrium!

CEEI Properties: Summary

CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional



CEEI Properties: Summary

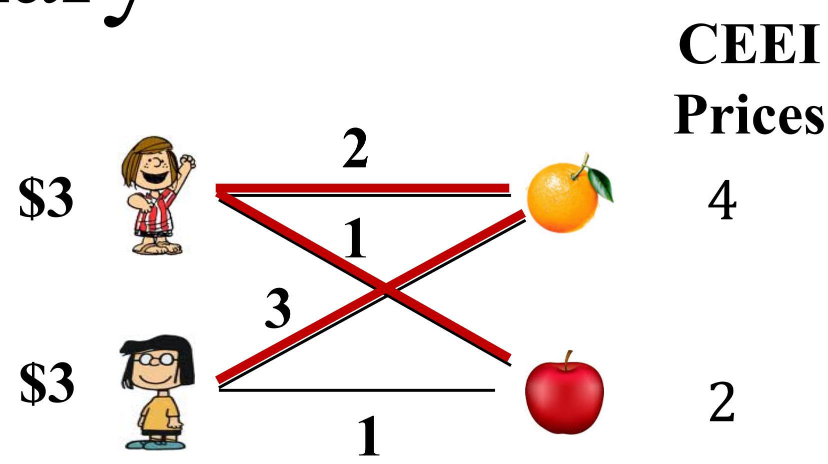
CEEI allocation is

- Pareto optimal (PO)
- Envy-free
- Proportional

$\forall i: \; v_i(x_i) \geq \frac{v_i(M)}{n}$

Next…

- **Nash welfare maximizing**



CEEI Allocation:

$$X_1 = \left(\frac{1}{4}, 1\right), X_2 = \left(\frac{3}{4}, 0\right)$$

$$V_1(X_1) = \frac{3}{2}, \; V_2(X_2) = \frac{9}{4}$$

$$V_1(X_2) = \frac{3}{2}, \; V_2(X_1) = \frac{7}{4}$$

Social Welfare

$$\sum_{i \in A} V_i(X_{i1}, \dots, X_{im})$$

Utilitarian

Issues: May assign 0 value to some agents.
Not scale invariant!

Max Nash Welfare

$$\textbf{max:} \prod_{i \in A} V_i(X_{i1}, \ldots, X_{im})$$

$$\textbf{s.t.} \quad \sum_{i \in A} X_{ij} \leq 1, \ \forall j \in G$$

$$X_{ij} \geq 0, \qquad \forall i, \forall j$$

Feasible allocations

Max Nash Welfare (MNW)

$$\textbf{max: } \log\left(\prod_{i\in A} V_i(X_{i1}, \ldots, X_{im})\right)$$

$$\textbf{s.t.} \quad \sum_{i\in A} X_{ij} \leq 1, \;\; \forall j \in G$$

$$X_{ij} \geq 0, \qquad \forall i, \forall j$$

Feasible allocations

Max Nash Welfare (MNW)

$$\textbf{max:} \sum_{i \in A} \log V_i(X_{i1}, \ldots, X_{im})$$

$$\textbf{s.t.} \quad \sum_{i \in A} X_{ij} \leq 1, \ \forall j \in G$$

$$X_{ij} \geq 0, \qquad \forall i, \forall j$$

Feasible allocations

Eisenberg-Gale Convex Program '59

max: $$\sum_{i \in A} \log V_i(\bar{X}_i)$$

Dual var.

s.t. $$\sum_{i \in A} X_{ij} \leq 1, \ \forall j \in G \longrightarrow p_j$$

$$X_{ij} \geq 0, \qquad \forall i, \forall j$$

Theorem. **Solutions of EG convex program are exactly the CEEI (p, X).**

Proof.

Consequences: CEEI

- **Exists**
- **Forms a convex set**
- **Can be *computed* in polynomial time**
- **Maximizes Nash Welfare**

Theorem. **Solutions of EG convex program are exactly the CEEI (p, X).**

Proof. **$\Rightarrow$ (Using KKT)**

Recall: CEEI Characterization

Pirces $p = (p_1, \ldots, p_m)$ and allocation $X = (X_1, \ldots, X_n)$

- Optimal bundle: For each buyer i
  - $p \cdot X_i = 1$
  - $X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = \max_{k \in M} \frac{V_{ik}}{p_k}$, for all good j

- Market clears: For each good j,

$$\sum_i X_{ij} = 1.$$

Theorem. **Solutions of EG convex program are exactly the CEE.**

$$\textbf{max:} \sum_{i \in A} \log(V_i(\bar{X}_i)) \quad \rightarrow \sum_j V_{ij} X_{ij}$$

Dual var.

$$\textbf{s.t.} \quad \sum_{i \in A} X_{ij} \leq 1, \ \forall j \in G \longrightarrow p_j \geq 0$$

$$X_{ij} \geq 0, \qquad \forall i, \forall j$$

Proof. $\Rightarrow$ (Using KKT)

$$\forall j, \ p_j > 0 \Rightarrow \sum_i X_{ij} = 1$$

Dual condition to X_{ij}:

$$\frac{V_{ij}}{V_i(X_i)} \leq p_j \Rightarrow \frac{V_{ij}}{p_j} \leq V_i(X_i) \Rightarrow p_j > 0 \Rightarrow$$ **market clears**

buy only MBB goods

$$X_{ij} > 0 \Rightarrow \frac{V_{ij}}{p_j} = V_i(X_i)$$

$$\sum_j V_{ij} X_{ij} = \left(\sum_j p_j X_{ij}\right) V_i(X_i)$$

$$\Rightarrow \sum_j p_j X_{ij} = 1$$

$\Rightarrow$ **optimal bundle**

Efficient (Combinatorial) Algorithms

Polynomial time

- Flow based [DPSV'08]
  - General exchange model (barter system) [DM'16, DGM'17, CM'18]
- Scaling + Simplex-like path following [GM.SV'13]

Strongly polynomial time

- Scaling + flow [O'10, V'12]
  - Exchange model (barter system) [GV'19]

We will discuss some of these if there is interest.

Application to Display Ads: Pacing Eq.

- Google Display Ads
  - Each advertiser has
    - Budget B_i. Value v_{ij} for keyword j
  - Pacing Eq.: $(\lambda_1, \ldots, \lambda_n) \in [0,1]^n$ s.t.
    - First price auction with bids $\lambda_i v_{ij}$
    - For each agent i, if $\lambda_i < 1$ then total payment $= B_i$, else $\leq B_i$
- Equivalent to Fisher market with quasi-linear utilities!