CS 580: Algorithmic Game Theory, Fall 2022
HW 3 (due on Tuesday, 1st November at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 40 points.
2. You can work on any homework in groups of ($\leq$) two. Submit only one assignment per group. First submit your solutions on Gradescope and you can add your group member after submission.
3. If you discuss a problem with another group then write the names of the other group's members at the beginning of the answer for that problem.
4. Please type your solutions if possible in Latex or doc whichever is suitable, and submit on Gradescope.
5. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
6. Except where otherwise noted, you may refer to lecture slides/notes. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you **cite them appropriately**, and make sure to **write in your own words**.
7. No late assignments will be accepted.
8. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (*Nash equilibrium: existence, computation, and complexity*)

   (a) (2 points) Find a Nash equilibrium of the game whose payoff bimatrix is given in Table 1.

5,3	8,0	-8,3
0,10	5,3	9,3
5,-10	5,11	5,3

Table 1: Payoff bimatrix of a 3X3 game

(b) (4 point) Is the problem of computing a Nash equilibrium scale invariant? That is, if (x, y) is a NE of game (A, B), then for $\lambda, \kappa \geq 0$ and $a, b \in \mathbb{R}$ is it also a NE of game $(\lambda A + a, \kappa B + b)$? Justify your answer.

What about ϵ-NE, where no player can gain more then ϵ by unilateral deviation. Is the problem of finding an ϵ-NE scale invariant? Justify your answer.

(c) (4 points) Prove that PPAD $\subseteq$ PPP . That is, show that the canonical problem of PPAD reduces to the canonical problem of PPP .

Recall that PPAD is the class of search problems reducible to EndOfLine , where the input to EndOfLine is two circuits N, P representing a possible next and previous node in an exponentially large graph where the vertices are in $\{0,1\}^n$ and, given that 0^n is a source, we need to return another vertex that is either a source or a sink. Also recall that PPP is the class of problems reducible to Pigeon , where the input to Pigeon is a circuit with equal number of input and output bits, and we need to either find a string $x \in \{0,1\}^n$ such that $C(x) = 0^n$, or two strings $x \neq y$ such that $C(x) = C(y)$.

2. (*Other game and equilibrium notions*)

(a) (2 points) Compute all Nash equilibria of the game shown in Table 2.

-1.4	1,-8	10, -2	3,2
3,-5	5,-2	-10, -9	5,-4
-3,-2	4,-5	-3, -5	8,-4
-2,1	4,1	9, -5	4,0

Table 2: Payoff bimatrix of a 4X4 game

[Hint: Apply iterated dominance.]

(b) (3 points) Alice and Bob are playing a game (A, B) in rounds where in t^{th} round they update their strategies as follows, starting at uniformly random strategies $x(0)$ and $y(0)$.

$$\begin{array}{rrcl} \forall i, & x_i(t) & = & x_i(t-1)\frac{(Ay)_i}{x^T Ay} \\ \forall j, & y_j(t) & = & y_j(t-1)\frac{(x^T B)_j}{x^T By} \end{array}$$

Assume that $A_{ij}, B_{ij} \geq 0$ for all i, j and that A, B are not identically zero matrices. Show that $(x(t), y(t)) = (x(t-1), y(t-1))$ if and only if $(x(t), y(t))$ is a Nash equilibrium.

(c) (3 points) Given a game (A, B), show that each of its correlated equilibria is also a coarse correlated equilibrium.

(d) (2 points) Write the normal form representation of the extensive form game with imperfect information shown in Figure 1.

3. (*Stackelberg strategies*) (10 points)

Consider the following Stackelberg game with three firms. Firm 1 chooses the quantity of its production first, then firms 2 and 3 choose their quantities simultaneously after observing firm 1's quantities. Suppose that they produce the same product with different cost functions. Firm 1's total cost is $C_1(q_1) = 10q_1 + 10$, firm 2's total cost is $C_2(q_2) = 8q_2$, and firm

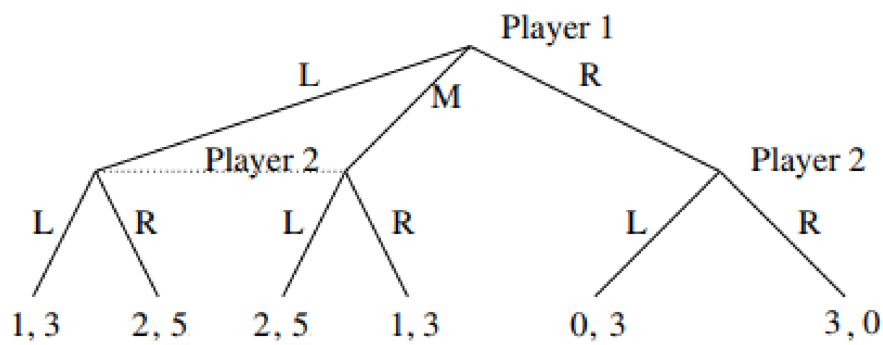

Figure 1: Extensive Form Game

3's total cost is $C_3(q_3) = 4q_3$. The firms produce identical goods and the market price is $P(q_1 q_2, q_3) = 300 - q_1 - q_2 - q_3$. What quantities do the firms produce in the subgame perfect equilibrium?

4. Consider a combinatorial auction with n bidders and n items where each bidder i has a unit-demand valuation v_i. This means that $v_i(S) = \max_{j \in S} v_{i,j}$ for every subset S of items. We assume that $v_{i,j} > 0$ for all i, j.

   In this auction, each bidder i submits one bid $b_{i,j}$ for each item j, and each item is sold separately using a second-price single-item auction. Assume that $b_{i,j} \in (0, v_{i,j}]$ for all i, j. The utility of a bidder is her value for the items won, minus her total payment. For example, if bidder i has values v_{i1} and v_{i2} for two items, and wins both items when the second-highest bids are p_1 and p_2, then her utility is $\max\{v_{i1}, v_{i2}\} - (p_1 + p_2)$. Let $G = (A, B)$ be a bipartite graph where A is the set of bidders and B is the set of items.

   - (2 points) Show that every allocation π of items to bidders that maximizes the Social Welfare $\left(\sum_i v_{i,\pi(i)}\right)$ induces a matching on G.
   - (8 points) Show that the PoA of PNE in such a game can be at most 2. Recall that the PoA is

   $$PoA = \max_{B=(b_1,\ldots,b_n):B \text{ is a NE}} \frac{OPT}{\sum_i \max_{j \in S^B(i)} v_{ij}} = \max_{B=(b_1,\ldots,b_n):B \text{ is a NE}} \frac{\sum_i v_{i,\pi(i)}}{\sum_i \max_{j \in S^B(i)} v_{ij}}$$

   where $S^B(i)$ is the set of items assigned to agent i at NE B, and π represents the optimal allocation.

5. (*Bonus problems*)

   (a) Prove that finding NE in game (A, B) reduces to finding a symmetric NE in a symmetric game.

   (b) The colorful Carathéodory theorem (CCT) is as follows. In d-dimensions, we are given points colored with a color from $\{1, \ldots, (d+1)\}$. Furthermore, let S_i be the set of points with color i, then $|S_i| = (d+1)$ and $\mathbf{0}$ is in the convex hull of S_i, for all i. We say that

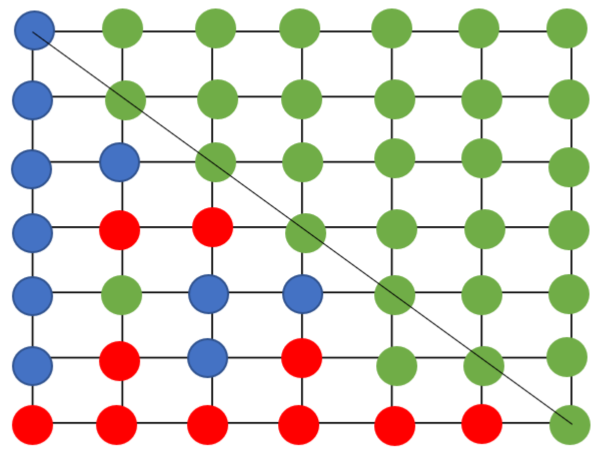

Figure 2: Illustrating the special case of Sperner

set S is colorful if it has exactly one point from each S_i. CCT proves that there exists a colorful set S whose convex hull contains $\mathbf{0}$.

Look up the proof of CCT and, using it, show that finding such a colorful set S is in PLS .

(c) Prove that checking if 1-D Sperner has more than one solution is NP-complete.

(d) Consider the special case of the 2-D Sperner problem on a square that adds the following further restrictions to the legal colorings of the boundary vertices. Every vertex on the upper side of the diagonal connecting the top-left and bottom right vertices, including all except the top-left vertex on the diagonal vertices must have the same color. Further, each vertex on the left boundary must have the same color as that of the top-left vertex, and each vertex on the bottom boundary must have the color assigned to the bottom-right vertex.

Figure 2 illustrates this special case. For clarity, most diagonal edges forming the triangles are not shown, except those on the main diagonal. Given that the 2-D Sperner problem we discussed in the class is PPAD -hard, prove that this special case of the problem is also PPAD -hard. That is, given an arbitrary 2-D Sperner instance reduce it to this special case of 2-D Sperner.