- A seller/auctioneer.
- Selling "single item".
- \( N \): set of agents/bidders/players.

\[ \text{agent } i \in N \text{ values the item at say } v_i \]

\[ \text{private info/true type } b_i \text{ to agent } i. \]

**Sealed Bid Auctions:**

1. Auctioneer solicits "bids" from the agents in a sealed envelope.

\[ \text{agent } i \text{ bids } b_i \text{ in a sealed envelope.} \]

\[ b_i \text{ need not be } v_i \]

2. Auctioneer opens all envelopes, looks at the bids' values

\[ i \text{ decides} \]

\[ \text{Auctioneer's Goal: max } \sum_i v_i \]

\[ \text{give the item to agent } i \text{ with } v_i \text{ max.} \]

\[ \text{winner = agent } i \text{ with } v_i \text{ max.} \]

\[ \text{payment } = P \]

\[ u_i(b_1, \ldots, b_n) = v_i - P \text{ if } i = i^* \text{ (winner)} \]

\[ = 0 \text{ otherwise.} \]

\[ \text{winner = highest bidder} \]

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<thead>
<tr>
<th>1</th>
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<td>( v_1 \mid )</td>
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<tbody>
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<td>800</td>
<td>2000</td>
</tr>
<tr>
<td>$b_i$</td>
<td>1495</td>
<td>600</td>
<td>180</td>
</tr>
</tbody>
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**First price auction:** Highest bidder wins, pays the bid.

**Example:** $N = 1, 2$, $V_1, V_2 \sim U[0, 1]$

Suppose: $b_2 = \frac{V_2}{2}$

$$
u_i(b_1, b_2) = (V_i - b_1) \Pr[b_1 \geq b_2] + 0 \Pr[b_1 < b_2]

= (V_i - b_1) \Pr[b_1 \geq V_2]

= (V_i - b_1) (2b_2)

$$

$$\begin{align*}
\mathbf{b}_1 &= \arg\max_{b_1} \nu_i(b_1, b_2) \\
&= \arg\max_{b_1} (V_i - b_1)(2b_2) \\
&= \arg\max_{b_1} 2V_i b_1 - 2b_1^2
\end{align*}$$

$$\frac{d}{db_1} (2V_i b_1 - 2b_1^2) = 0$$

$$\Rightarrow 2V_i - 4b_1 = 0$$

$$\Rightarrow b_1 = \frac{V_i}{2}$$

Similarly, if $b_1 = \frac{V_i}{2}$ then best bid for agent 2 is $b_2 = \frac{V_2}{2}$.


\[ b_2 = \frac{v_0}{2} \]

\[ \left( \frac{v_1}{2}, \frac{v_2}{2} \right) \text{ is a BNE.} \]

Generalize, \( N = \{1, \ldots, n\} \), \( v_i \sim U[0, 1] \),

\[ b_i = \frac{(n-1)}{n} v_i \quad \forall i \quad \text{is a NE.} \]

1. What if \( v_i \) 's have a complex distribution?
2. " " " " " " different " " " " ?
3. What if agents are not fully rational?
4. If there are other NE & agents could not coordinate on what to play?

* Second Price: Highest bidder wins, pays the second highest bid.

\[ \text{winner} = \arg \max_i b_i = i^* \]

\[ b_i = \text{critical bid} \quad \forall i \neq i^* \]

\[ \text{payment } b_i = \begin{cases} \max_k b_k & \text{if } i = i^* \\ 0 & \text{otherwise} \end{cases} \]

\[ = 0 \quad \text{otherwise} \]

11. (Vickrey '61): Under second price, for each \( i \),
Thm (Vickery’61): Under second price, for each $i$, $b_i = v_i$ is an optimal bid no matter what others are bidding.

1)

$v_i$, $b_i = v_i$ is a DSE.

2)

$v_i$, $\forall b_i$, $u_i(v_i, b_i) \geq u_i(b_i, b_i)$, $\forall b_i$

**Proof**: Fix an agent $i$, fix $b_i$ (arbitrarily), agent $i$ bids $b_i$

$v_i = \max_{b_i} u_i(b_i)$

\[
\begin{cases}
\text{case I: } i \text{ wins.} & \Rightarrow \max_{k \neq i} b_k \\
\quad u_i(b_i, b_i) = v_i - b_i & (b_i \geq B_i) \\
\text{case II: } i \text{ looses.} & \\
\quad u_i(b_i, b_{-i}) = 0.
\end{cases}
\]

\[u_i(b_i, b_{-i}) = u_i(v_i, b_{-i})\]

**Case I**

\[0 \leq u_i(v_i, b_i) \quad b_i\]

**Case II**

\[u_i(b_i, b_i) = 0 = u_i(v_i, b_{-i})\]
\( u_i(b_i, b_{-i}) = 0 = u_i(v_i, b_{-i}) \)

Dominant strategy Invative Compatible (DSIC)

Truthful Auction

"Ebay = second price."

English Auction: Increasing price.

\[ V_1 < V_2 \ldots \quad V_{m-1} \quad V_m \]

\[ p = 0 \]

Dutch Auction: Decreasing price

\[ V_1 \quad \ldots \quad V_m \]

\[ p = 1 \]

First price.