So far

- Normal-form games
  - Multiple rational players, single shot, simultaneous move
- Bayesian Games (Incomplete Information)

- Nash equilibrium
  - Existence
  - Computation in two-player games.
Today:

- Issues with NE
  - Multiplicity
  - Selection: How players decide/reach any particular NE

- Possible Solutions
  - Dominance: Dominant Strategy equilibria
  - Arbitrator/Mediator: Correlated equilibria, Coarse-correlated equilibria
  - Communication/Contract: Stackelberg equilibria, Nash bargaining

- Other Games
  - Extensive-form Games
Dominance

- **Strict dominance:** For a player, move $s$ strictly dominates $t$ if no matter what others play, $s$ gives her better payoff than $t$
  - for all $s_{-i}$, $u_i(s, s_{-i}) > u_i(t, s_{-i})$

- **Weakly dominates:** $s$ weakly dominates $t$ if
  - for all $s_{-i}$, $u_i(s, s_{-i}) \geq u_i(t, s_{-i})$; and
  - for some $s_{-i}$, $u_i(s, s_{-i}) > u_i(t, s_{-i})$

$$-i = "the player(s) other than i"$$

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>U</strong></td>
<td>0, 0</td>
<td>1, -1</td>
<td>1, -1</td>
</tr>
<tr>
<td><strong>G</strong></td>
<td>-1, 1</td>
<td>0, 0</td>
<td>-1, 1</td>
</tr>
<tr>
<td><strong>B</strong></td>
<td>-1, 1</td>
<td>1, -1</td>
<td>0, 0</td>
</tr>
</tbody>
</table>
Dominant Strategy Equilibrium

Playing move $s_i$ is best for agent $i$, no matter what others play.

For each player $i$, there is a (strategy) move $s_i$ that (weakly) dominates all other moves.

- for all $i, s'_i, s_{-i}$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$;

Example?
Prisoner’s Dilemma

- Pair of criminals has been caught
- They have two choices: {confess, don’t confess}

<table>
<thead>
<tr>
<th></th>
<th>confess</th>
<th>don’t confess</th>
</tr>
</thead>
<tbody>
<tr>
<td>confess</td>
<td>-5, -5</td>
<td>0, -6</td>
</tr>
<tr>
<td>don’t confess</td>
<td>-6, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>
“Should I buy an SUV?”

- Purchasing cost:
  - SUV: 5
  - Sedan: 3

- Accident cost:
  - SUV: 8
  - Sedan: 2

<table>
<thead>
<tr>
<th></th>
<th>-10, -10</th>
<th>-7, -11</th>
</tr>
</thead>
<tbody>
<tr>
<td>-11, -7</td>
<td>-8, -8</td>
<td></td>
</tr>
</tbody>
</table>
Dominance by Mixed strategies

Example of dominance by a mixed strategy:

\[
\begin{array}{|cc|}
\hline
3, 1 & 0, 0 \\
\hline
0, 0 & 3, 2 \\
\hline
1, 0 & 1, 1 \\
\hline
\end{array}
\]
Iterated dominance: path (in)dependence

Iterated **weak dominance** is **path-dependent**: sequence of eliminations may determine which solution we get (if any) (whether or not dominance by mixed strategies allowed)

<table>
<thead>
<tr>
<th></th>
<th>0,1</th>
<th>0,0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,0</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>0,1</td>
<td>0,0</td>
<td></td>
</tr>
</tbody>
</table>

Iterated **strict dominance** is **path-independent**: elimination process will always terminate at the same point (whether or not dominance by mixed strategies allowed)

<table>
<thead>
<tr>
<th></th>
<th>0,1</th>
<th>0,0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,0</td>
<td>1,0</td>
<td>1,0</td>
</tr>
<tr>
<td>0,1</td>
<td>0,0</td>
<td></td>
</tr>
<tr>
<td>0,0</td>
<td>0,1</td>
<td></td>
</tr>
</tbody>
</table>
**NE:** \( x^T Ay \geq x'^T Ay, \ \forall x' \quad x^T By \geq x^T B y', \ \forall y' \)

No one plays dominated strategies.

Why?

What if they can discuss beforehand?
Players: \{Alice, Bob\}

Two options: \{Football, Tennis\}

At Mixed NE both get \(\frac{2}{3} < 1\)

Instead they agree on \(\frac{1}{2}(F, T), \frac{1}{2}(T, F)\)

Payoffs are (1.5, 1.5)  
Fair!

Needs a common coin toss!
Correlated Equilibrium – (CE)
(Aumann’74)

- **Mediator** declares a joint distribution $P$ over $S = \times_i S_i$
- Tosses a coin, chooses $s = (s_1, \ldots, s_n) \sim P$.
- Suggests $s_i$ to player $i$ in private

- $P$ is at equilibrium if each player wants to follow the suggestion when others do.
  
  $U_i(s_i, P_{(s_{-i})}) \geq U_i(s_i', P_{(s_{-j})}), \forall s_i' \in S_1$
CE for 2-Player Case

- **Mediator** declares a joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$

- Tosses a coin, chooses $(i, j) \sim P$.
- Suggests $i$ to Alice, $j$ to Bob, in private.

- $P$ is a CE if each player wants to follow the suggestion, when the other does.

Given Alice is suggested $i$, she knows Bob is suggested $j \sim P(i, .)$

\[
\begin{align*}
\langle A(i, .), P(i, .) \rangle &\geq \langle A(i', .), P(i, .) \rangle : \forall i' \in S_1 \\
\langle B(., j), P(., j) \rangle &\geq \langle B(., j'), P(., j) \rangle : \forall j' \in S_2
\end{align*}
\]
Players: \{Alice, Bob\}

Two options: \{Football, Shopping\}

Instead they agree on \(\frac{1}{2}(F, S), \frac{1}{2}(S, F)\)

Payoffs are \((1.5, 1.5)\) CE! Fair!
### Prisoner’s Dilemma

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>NC</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>-5, -5</td>
<td>0, -6</td>
</tr>
<tr>
<td>NC</td>
<td>-6, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

NC is dominated

### Rock-Paper-Scissors (Aumann)

<table>
<thead>
<tr>
<th></th>
<th>R</th>
<th>P</th>
<th>S</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0, 0</td>
<td>0, 1</td>
<td>1, 0</td>
</tr>
<tr>
<td>P</td>
<td>1, 0</td>
<td>0, 0</td>
<td>0, 1</td>
</tr>
<tr>
<td>S</td>
<td>0, 1</td>
<td>1, 0</td>
<td>0, 0</td>
</tr>
</tbody>
</table>

When Alice is suggested R
Bob must be following $P_{(R,.)} \sim (0, 1/6, 1/6)$
Following the suggestion gives her 1/6
While P gives 0, and S gives 1/6.
Computation: Linear Feasibility Problem

Game $(A, B)$. Find, joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$

\[
\begin{align*}
\sum_j A_{ij} p_{ij} & \geq \sum_j A_{ij} p_{ij} \quad \forall i, i' \in S_1 \\
\sum_i B_{ij} p_{ij} & \geq \sum_i B_{ij} p_{ij} \quad \forall j, j' \in S_2 \\
\sum_{ij} p_{ij} & = 1; \quad p_{ij} \geq 0, \quad \forall (i, j)
\end{align*}
\]

$N$-player game: Find distribution $P$ over $S = \times_{i=1}^N S_i$

\[
\begin{align*}
\sum_{s' \in S_i} U_i(s_i, P_{(s_i,.)}) & \geq U_i(s'_i, P_{(s_i,.)}), \quad \forall s_i, s'_i \in S_i \\
\sum_{s \in S} P(s) & = 1 \\
\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) P(s_i, s_{-i}) & \quad \text{Linear in } P \text{ variables!}
\end{align*}
\]
Computation: Linear Feasibility Problem

N-player game: Find distribution $P$ over $S = \times_{i=1}^{N} S_i$

s.t. $U_i(s_i, P_{(i,\cdot)}) \geq U_i(s'_i, P_{(s_i,\cdot)}), \forall s_i, s'_i \in S_i$

$\sum_{s \in S} P(s) = 1$

$\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i})P(s_i, s_{-i})$  Linear in $P$ variables!

Can optimize any convex function as well!
Coarse-Correlated Equilibrium

- After mediator declares \( P \), each player opts in or out.
- Mediator tosses a coin, and chooses \( s \sim P \).
- If player \( i \) opted in, then the mediator suggests her \( s_i \) in private, and she has to obey.
- If she opted out, then (knowing nothing about \( s \)) plays a fixed strategy \( t \in S_i \).
- At equilibrium, each player wants to opt in, if others are.

\[ U_i(P) \geq U_i(t, P_{-i}), \quad \forall t \in S_i \]

Where \( P_{-i} \) is joint distribution of all players except \( i \).
Importance of (Coarse) CE

- Natural dynamics quickly arrive at approximation of such equilibria.
  - No-regret, Multiplicative Weight Update (MWU)

- Poly-time computable in the size of the game.
  - Can optimize a convex function too.
Show the following
Extensive-form Game

- Players move one after another
  - Chess, Poker, etc.
  - Tree representation.

Strategy of a player:
What to play at each of its node.

<table>
<thead>
<tr>
<th></th>
<th>I</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>F</td>
<td>-1, -1</td>
<td>2, 0</td>
</tr>
<tr>
<td>A</td>
<td>1, 1</td>
<td>2, 0</td>
</tr>
</tbody>
</table>

Entry game
A poker-like game

- Both players put 1 chip in the pot
- Alice gets a card (King is a winning card, Jack a losing card)
- Alice decides to raise (add one to the pot) or check
- Bob decides to call (match) or fold (Alice wins)
- If Bob called, he adds one to the pot. Alice’s card determines pot winner.
Poker-like game in normal form

Can be exponentially big!
Sub-Game Perfect Equilibrium

- Every sub-tree is at equilibrium
- Computation when perfect information (no nature/chance move): **Backward induction**

![Game Tree Image]

**Entry game**

New Firm

Old Firm

- out
- in

2,0

- fight
- accommodate

-1,-1

1,1

accommodate

2,0

2,0

1,1
Sub-Game Perfect Equilibrium

- Every sub-tree is at equilibrium
- Computation when perfect information (no nature/chance move): **Backward induction**

Entry game

New Firm

Old Firm

(accommodate, in)

New Firm

out

2,0

in

1,1

accommodate

2,0

in

0,2

fight

-1,-1

accommodate

1,1
Corr. Eq. in Extensive form Game

- How to define?
  - CE in its normal-form representation.
- Is it computable?
  - Recall: exponential blow up in size.
- Can there be other notions?

Commitment
(Stackelberg strategies)
Suppose the game is played as follows:

- Alice commits to playing one of the rows,
- Bob observes the commitment and then chooses a column

Optimal strategy for Alice: commit to Down

Unique Nash equilibrium (iterated strict dominance solution)
Commitment: an extensive-form game

For the case of committing to a pure strategy:

```
Player 1
(Alice)

Up          Down

Player 2
(Bob)

Left       Right       Left       Right

1, 1   3, 0  0, 0   2, 1
```
Commitment to mixed strategies

Also called a Stackelberg (mixed) strategy
Commitment: an extensive-form game

- ... for the case of committing to a mixed strategy:

- Economist: Just an extensive-form game, nothing new here
- Computer scientist: Infinite-size game! Representation matters
Computing the optimal mixed strategy to commit to
[Conitzer & Sandholm EC’06]

- Player 1 (Alice) is a leader.
- Separate LP for Bob’s move (column) $j^* \in S_2$:

\[
\begin{align*}
\text{maximize } & \sum_i x_i A_{ij^*} \\
\text{subject to } & \forall j, \ (x^T B)_{j^*} \geq (x^T B)_j \\
& x \geq 0, \ \sum_i x_i = 1
\end{align*}
\]

Alice’s utility when Bob plays $j^*$
Playing $j^*$ is best for Bob
$x$ is a probability distribution

Among soln. of all the LPs, pick the one that gives max utility.
On the game we saw before

\[
\begin{array}{ccc}
 & 1, 1 & 3, 0 \\
 x_1 & 0, 0 & 2, 1 \\
 x_2 & & \\
\end{array}
\]

\[
\begin{align*}
\text{maximize} & \quad 1x_1 + 0x_2 \\
\text{subject to} & \quad 1x_1 + 0x_2 \geq 0x_1 + 1x_2 \\
& \quad x_1 + x_2 = 1 \\
& \quad x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{maximize} & \quad 3x_1 + 2x_2 \\
\text{subject to} & \quad 0x_1 + 1x_2 \geq 1x_1 + 0x_2 \\
& \quad x_1 + x_2 = 1 \\
& \quad x_1 \geq 0, x_2 \geq 0
\end{align*}
\]
Visualization

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0,1</td>
<td>1,0</td>
<td>0,0</td>
</tr>
<tr>
<td>M</td>
<td>4,0</td>
<td>0,1</td>
<td>0,0</td>
</tr>
<tr>
<td>D</td>
<td>0,0</td>
<td>1,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>
Generalizing beyond zero-sum games

Minimax, Nash, Stackelberg all agree in zero-sum games

zero-sum games

minimax strategies

Nash equilibrium

zero-sum games  general-sum games

Stackelberg mixed strategies
Other nice properties of commitment to mixed strategies

• No equilibrium selection problem

• Leader’s payoff at least as good as any Nash eq. or even correlated eq.

(von Stengel & Zamir [GEB ‘10])
Nash Bargaining
Nash Bargaining: Dividing Utilities

Two agents: 1, 2
Outside option utilities: $c_1, c_2$
Feasible set of Utilities: $U \subseteq \mathbb{R}^2$ (convex),
\[(c_1, c_2) \in U\]

**Goal:** define a bargaining function $f(c_1, c_2, U) \in U$
Satisfying certain good properties
Nash Bargaining: Axioms

Two agents: 1, 2
Outside option with utilities: $c_1, c_2$
Feasible set of Utilities: $U \subseteq R^2$ (convex), $(c_1, c_2) \in U$

**Goal:** $f(c_1, c_2, U) \in U$ that is
1. Scale free
2. Symmetric
3. Pareto Optimal
4. Independent of Irrelevant Alternatives (IIA)
5. Individually Rational
Nash Bargaining: Theorem

Two agents: 1, 2
Outside option with utilities: \( c_1, c_2 \)
Feasible set of Utilities: \( U \subseteq R^2 \) (convex), \( (c_1, c_2) \in U \)

**Goal:** \( f(c_1, c_2, U) \in U \) that is

1. Scale free
2. Symmetric
3. Pareto Optimal
4. Independent of Irrelevant Alternatives (IIA)
5. Individually Rational

**Theorem (Nash’50).** \( f \) satisfies the 5 axioms if and only if, \( f(c_1, c_2, U) \) is

\[
\arg \max (u_1 - c_1)(u_2 - c_2) \\
\text{s.t.} \quad (u_1, u_2) \in U
\]
**Nash Bargaining: Theorem**

**Theorem (Nash’50).** \( f \) satisfies the 5 axioms if and only if, 
\[
  f(c_1, c_2, U) \text{ is } \arg\max_{(u_1, u_2) \in U} (u_1 - c_1)(u_2 - c_2)
\]

**Proof.** (\(\Leftarrow\))
1. Scale free
2. Symmetric
3. Pareto Optimal
4. Independent of Irrelevant Alternatives (IIA)
5. Individually Rational