CS 580: Algorithmic Game Theory, Fall 2021
HW 0

Instructions:

1. The purpose of this homework is to reacquaint you with topics, ideas and tools that will be needed in this course.

2. This homework is NOT for submission and will NOT be graded.

1. Let \( a_1, a_2, \ldots, a_n \) be fixed real numbers and \( X \) be a random variable that takes value \( a_i \) with some probability \( p_i \). Define the set of probability distributions that maximize \( E[X] \).

2. Consider throwing \( n \) balls into \( n \) bins where each ball is thrown independently and uniformly at random into a bin.

   (a) What is the probability that a given bin (say the first bin) is empty?

   (b) What is the probability that it contains exactly \( k \) balls?

   (c) What is the expected number of bins that are empty?

3. Consider the following linear program:

\[
\begin{align*}
\max & \quad c^T x \\
\text{s.t.} & \quad Ax \leq b \\
& \quad x \geq 0
\end{align*}
\]

   (a) Write the dual linear program of the above LP.

   (b) Write the corresponding complementary slackness conditions.

   (c) Using the complementary slackness conditions, derive the strong duality theorem.

   \( (\text{If } x^* \text{ is an optimal solution to the primal LP and } y^* \text{ is an optimal solution to the dual LP, then } c^T x^* = b^T y^*. ) \)

4. A wheel of size \( k \) consists of a cycle on \( k \) vertices along with an additional vertex connected to every vertex in the cycle. As an example, you can see a wheel of size 8 in the figure below. The WHEEL problem is the following: Given an undirected graph \( G = (V, E) \) and an integer \( k \), does \( G \) contain a wheel of size \( k \) as a subgraph? Prove that WHEEL is NP-Complete.

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\text{(Hint: To show NP-hardness reduce from the Hamiltonian cycle problem.)} 
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