Prophet Inequalities
A Crash Course

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EC18: ACM CONFERENCE ON ECONOMICS AND COMPUTATION MENTORING WORKSHOP, JUNE 18, 2018
Profit
From Wikipedia, the free encyclopedia

Not to be confused with Prophet.

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Not to be confused with Profit.
The Plan

1. Introduction to Prophet Inequalities
2. Connections to Pricing and Mechanism Design
Prophet Inequality

The gambler’s problem:

\[ D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \]
Prophet Inequality

The gambler’s problem:

Keep: win $20, game stops.
Discard: prize is lost, game continues with next box.
Let’s Play...

\[
\begin{align*}
3.16 & \quad U[2,4] \\
2.87 & \quad U[2,4] \\
1.14 & \quad U[1,5] \\
2.67 & \quad U[0,7]
\end{align*}
\]
Prophet Inequality

**Theorem:** [Krengel, Sucheston, Garling ‘77]

There exists a strategy for the gambler such that

\[ E[\text{prize}] \geq \frac{1}{2} E \left[ \max_i v_i \right] \]

and the factor 2 is tight.

[Samuel-Cahn ‘84] ... a fixed threshold strategy: choose a single threshold \( t \), accept first prize \( \geq t \).
Lower Bound: 2 is Tight

\[ E \left[ \max_i \tau_i \right] = 1(1-\varepsilon) + \frac{1}{\varepsilon} \cdot \varepsilon \sim 2-\varepsilon \]
Theorem: [Samuel-Cahn ‘84]

Given distributions $G_1, \ldots, G_n$ where $\pi_i \sim G_i$, there exists a fixed threshold strategy (accept first prize $\geq t$) such that

$$E_{\pi}[\text{prize}] \geq \frac{1}{2} E_{\pi} \left[ \max_i \pi_i \right]$$

Proof:
Application: Posted Pricing

A mechanism design problem:
1 item to sell, n buyers, independent values \( v_i \sim D_i \).
Buyers arrive sequentially, in an arbitrary order.
For each buyer: interact according to some protocol, decide whether or not to trade, and at what price.

Corollary of Prophet Inequality:
Posting an appropriate take-it-or-leave-it price \( t \) yields at least half of the expected optimal social welfare.

[Hajiaghayi Kleinberg Sandholm ’07]
Applications

What about revenue?

[Chawla Hartline Malec Sivan ’10]: Can apply prophet inequality to virtual values to achieve half of optimal revenue.

\[
E[Rev] = E_v \left[ \sum_i p_i(v) \right] = E_v \left[ \sum_i \phi_i(v_i)x_i(v) \right]
\]

(for single item)

\[
= E_v [\max_i \phi_i(v_i)^+]
\]

Auction w/ \( E[Rev] \geq \frac{1}{2} OPT \)

1. Distribution \( G_i \) on \( \phi_i(v_i)^+ \) using \( F_i \) on \( v_i \)
2. Compute \( t \) s.t. \( \Pr \left[ \max_i \phi_i(v_i)^+ \geq t \right] = 1/2 \) (t s.t. Prob. Of selling is \( \frac{1}{2} \))
3. Give to an agent with \( \phi_i(v_i)^+ \geq t \)
   • With highest value
4. Payment = \( \max \{ \phi_i^{-1}(t), \text{second highest bid} \} \)
Alternate Pricing

Multiple choices of $p$ that achieve the 2-approx of total value. Here’s one due to [Kleinberg Weinberg 12]:

Theorem (prophet inequality): for one item, setting threshold $p = \frac{1}{2}E \left[ \max_i v_i \right]$ yields expected welfare $\geq \frac{1}{2}E \left[ \max_i v_i \right]$.

Example:

1 or 6  0 or 8  2 or 10

(each box: prizes equally likely)

$\text{OPT} = \begin{cases} 10 & \text{w.p. 1/2} \\ 8 & \text{w.p. 1/4} \\ 6 & \text{w.p. 1/8} \\ 2 & \text{w.p. 1/8} \end{cases}$

$E[\text{OPT}] = 8 
\rightarrow \text{accept first prize} \geq 4$
Prophet Inequality: Proof

Theorem (prophet inequality): for one item, setting threshold
\[ p = \frac{1}{2} E \left[ \max_i v_i \right] \] yields expected value \( \geq \frac{1}{2} E \left[ \max_i v_i \right] \).

What can go wrong?

If threshold is

- **Too low**: we might accept a small prize, preventing us from taking a larger prize in a later round.
- **Too high**: we don’t accept *any* prize.
A Proof for Full Information

Case 1: Somebody $i < i^*$ buys the item.

$\Rightarrow$ revenue $\geq \frac{1}{2} v_{i^*}$

Case 2: Nobody $i < i^*$ buys the item.

$\Rightarrow$ utility of $i^* \geq v_{i^*} - \frac{1}{2} v_{i^*} = \frac{1}{2} v_{i^*}$

In either case: welfare $= \text{revenue} + \text{buyer utilities} \geq \frac{1}{2} v_{i^*}$
Extending to Stochastic Setting

**Thm:** setting price \( p = \frac{1}{2} E \left[ \max_i v_i \right] \) yields value \( \geq \frac{1}{2} E \left[ \max_i v_i \right] \).

**Proof.** Random variable: \( v^* = \max_i v_i = OPT \)

1. \( \text{REVENUE} = p \cdot \Pr[\text{item is sold}] = \frac{1}{2} E[v^*] \cdot \Pr[\text{item is sold}] \)
2. \( \text{SURPLUS} = \sum_i E[\text{utility of buyer } i] \)
   \( \geq \sum_i E[(v_i - p)^+ \cdot 1[i \text{ sees item}]] \)
   \( = \sum_i E[(v_i - p)^+] \cdot \Pr[i \text{ sees item}] \)
   \( \geq \sum_i E[(v_i - p)^+] \cdot \Pr[i \text{ item not sold}] \)
   \( \geq E \left[ \max_i (v_i - p) \right] \cdot \Pr[i \text{ item not sold}] \)
   \( \geq \frac{1}{2} E[v^*] \cdot \Pr[i \text{ item not sold}] \)
3. Total Value = \text{REVENUE} + \text{SURPLUS} \geq \frac{1}{2} E[v^*]. \)
Prophet Inequality: Proof

Thm: for one item, price \( p = \frac{1}{2} E[OPT] \) yields value \( \geq \frac{1}{2} E[OPT] \).

Summary:

• Price is high enough that expected revenue offsets the opportunity cost of selling the item.
• Price is low enough that expected buyer surplus offsets the value left on the table due to the item going unsold.
Secretaries and Prophet Secretaries
A Variation

Prophet Inequality:
Prizes drawn from distributions, order is arbitrary

A Related Problem:
Prizes are arbitrary, order is uniformly random
Let’s Play...

The game of googol [Gardner ‘60]
Secretary Problem

**Theorem:** [Lindley ’61, Dynkin ‘63, Gilbert and Mosteller ‘66]

There exists a strategy for the secretary problem such that

\[ \Pr[\text{select largest}] \geq \frac{1}{e} \]

and the factor \( e \) is tight as \( n \) grows large.

**Strategy:** observe the first \( n/e \) values, then accept the next value that is larger than all previous.
Prophets vs Secretaries

**Prophet Inequality:**
Prizes drawn from distributions, order is arbitrary

**Secretary Problem / Game of Googol:**
Prizes are arbitrary, order is uniformly random

**Prophet Secretary:**
Prizes drawn from distributions, order is uniformly random and revealed online

[Esfandiari, Hajiaghayi, Liaghat, Monemizadeh ‘15]
Recall:

\[
U[2,4] \quad U[2,4] \quad U[1,5] \quad U[0,7]
\]
Recall:

\[ U[0,7] \quad U[1,5] \quad U[2,4] \quad U[2,4] \]
Theorem: [Esfandiari, Hajiaghayi, Liaghat, Monemizadeh ‘15]

There exists a strategy for the gambler such that

$$E[\text{prize}] \geq \left(1 - \frac{1}{e}\right) E \left[\max_i v_i\right].$$

[Azar, Chiplunkar, Kaplan EC’18]: A strategy for the gambler that beats \(\left(1 - \frac{1}{e}\right).\)
Prophet Secretary

threshold

value

prize

round
Prophet Secretary

Higher threshold:
more revenue when we sell the item to this buyer.

Lower threshold:
More surplus for this buyer.
Extension: Multiple Prizes
Multiple-Prize Prophet Inequality

Prophet inequality, but gambler can keep up to $k$ prizes

$k = 1$: original prophet inequality: 2-approx

$k \geq 1$: [Hajiaghayi, Kleinberg, Sandholm ‘07]  
There is a threshold $p$ such that picking the first $k$ values $\geq p$  
gives a $1 + O(\sqrt{\log k/k})$ approximation.

**Idea:** choose $p$ s.t. expected # of prizes taken is $k - \sqrt{2k \log k}$.  
Then w.h.p. # prizes taken lies between $k - \sqrt{4k \log k}$ and $k$.

[Alaei ‘11] [Alaei Hajiaghayi Liaghat ‘12] Can be improved to  
$1 + O\left(\frac{1}{\sqrt{k}}\right)$ using a randomized strategy, and this is tight.
## Aside: Beyond Cardinality

<table>
<thead>
<tr>
<th>Constraint</th>
<th>Upper Bound</th>
<th>Lower Bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>Single item</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$k$ items</td>
<td>$1 + O\left(\frac{1}{\sqrt{k}}\right)$</td>
<td>$1 + \Omega\left(\frac{1}{\sqrt{k}}\right)$</td>
</tr>
<tr>
<td>Matroid</td>
<td>$\frac{2}{k}$ [Kleinberg Weinberg ‘12]</td>
<td>$\frac{2}{k}$ [Kleinberg Weinberg ‘12]</td>
</tr>
<tr>
<td>$k$ matroids</td>
<td>$e \cdot (k + 1)$ [Feldman Svensson Zenklusen ‘15]</td>
<td>$\sqrt{k} + 1$ [Kleinberg Weinberg ‘12]</td>
</tr>
<tr>
<td>Knapsack</td>
<td>5 [Duetting Feldman Kesselheim L. ‘17]</td>
<td>2</td>
</tr>
<tr>
<td>Downward-closed, max set size $\leq r$</td>
<td>$O(\log n \log r)$ [Rubinstein ‘16]</td>
<td>$\Omega\left(\frac{\log n}{\log \log n}\right)$ [Babaioff Immorlica Kleinberg ‘07]</td>
</tr>
</tbody>
</table>

Directly imply posted-price mechanisms for welfare, revenue
Multiple-Prize Prophet Inequality

A different variation on cardinality:

- The gambler can choose up to $k \geq 1$ prizes
- Afterward, gambler can keep the *largest* of the prizes chosen

**Theorem** [Assaf, Samuel-Cahn ‘00]: There is a strategy for the gambler such that $E[\text{prize}] \geq \left(1 - \frac{1}{k+1}\right) E\left[\max_i v_i\right]$

[Ezra, Feldman, Nehama EC’18]: An extension to settings where gambler can *choose up to* $k$ prizes and *keep up to* $\ell$. Includes an improved bound for $\ell = 1$!
Combinatorial Variants

More general valuation functions:

Reward for accepting a set of prizes $S$ is a function $f(S)$. Example: arbitrary submodular. [Rubinstein, Singla ’17]

Multiple prizes per round:

Multiple boxes arrive each round. Revealed in round $i$: valuation function $f_i(S)$ for accepting set of prizes $S_i$ on round $i$. (Note: possible correlation!)

Application: posted-price mechanisms for selling many goods [Alaei, Hajiaghayi, Liaghat ‘12], [Feldman Gravin L ‘13], [Duetting Feldman Kesselheim L ’17]
Summary

• Prophet Inequalities: analyzing the power of sequential decision-making, vs an offline benchmark.
• Recent connections to pricing and mechanism design
• MANY variations! A very active area of research

Open Challenge: Best-Order Prophet Inequality
Suppose the gambler can choose which order to open boxes.
• What fraction of $E \left[ \max_i v_i \right]$ can the gambler guarantee?
• Can the best order be computed efficiently?

Thanks!
Bonus: Multi-Dimensional Prophets
A General Model

Combinatorial allocation

• Set $M$ of $m$ resources (goods)
• $n$ buyers, arrive sequentially online
• Buyer $i$ has valuation function $v_i: 2^M \rightarrow R_{\geq 0}$
• Each $v_i$ is drawn indep. from a known distribution $D_i$
• Allocation: $x = (x_1, \ldots, x_n)$.
  There is a downward-closed set $F$ of feasible allocations.

Goal: feasible allocation maximizing $\sum_i v_i(x_i)$
Posted Price Mechanism

1. For each bidder in some order $\pi$:
2. Seller chooses prices $p_i(x_i)$
3. Bidder $i$’s valuation is realized: $v_i \sim F_i$
4. $i$ chooses some $x_i \in \arg \max \{v_i(x_i) - p_i(x_i)\}$

Notes:
• “Obviously” strategy proof [Li 2015]
• Tie-breaking can be arbitrary
• Prices: static vs dynamic, item vs. bundle
• Special case: oblivious posted-price mechanism (OPM)
  prices chosen in advance, arbitrary arrival order
## Applications

<table>
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<tr>
<th>Problem</th>
<th>Approx.</th>
<th>Price Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>Combinatorial auction, XOS valuations</td>
<td>2</td>
<td>Static item prices</td>
</tr>
<tr>
<td>Bounded complements (MPH-k) [Feige et al. 2014]</td>
<td>$4k - 2$</td>
<td>Static item prices</td>
</tr>
<tr>
<td>Submodular valuations, matroid constraints</td>
<td>2 (existential) 4 (polytime)</td>
<td>Dynamic prices</td>
</tr>
<tr>
<td>Knapsack constraints</td>
<td>5</td>
<td>Static prices</td>
</tr>
<tr>
<td>d-sparse Packing Integer Programs</td>
<td>$8d$</td>
<td>Static prices</td>
</tr>
</tbody>
</table>

[Feldman Gravin L ‘13], [Duetting Feldman Kesselheim L ’17]