Lecture 9
PPAD and other TFNP classes

CS580

Ruta Mehta

Most slides are borrowed from Prof. C. Daskalakis's presentation.

Menu

↳ Existence Theorems: **Nash**, Brouwer, Sperner

Games and Equilibria

		2/5	3/5
	Kick / Dive	Left	Right
1/2	Left	2 , -1	-1 , 1
1/2	Right	-1 , 1	1, -1

Equilibrium:

A pair of randomized strategies so that no player has incentive to deviate if the other stays put.

[Nash ’50]: *An equilibrium exists in every game.*

no poly-time algorithm known, despite intense effort

Menu

→ Existence Theorems: Nash, **Brouwer,** Sperner

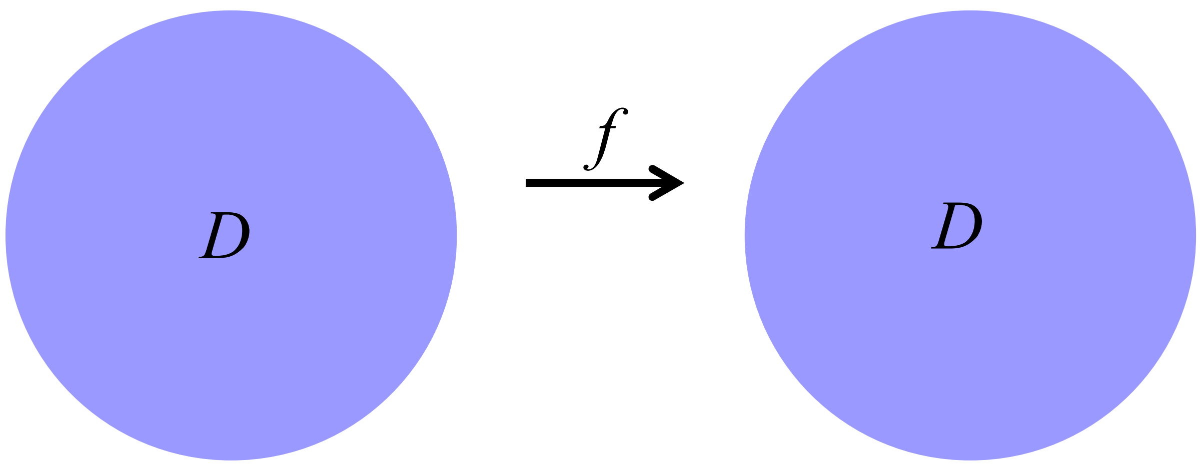
Brouwer's Fixed Point Theorem

[Brouwer 1910]: Let $f : D \to D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

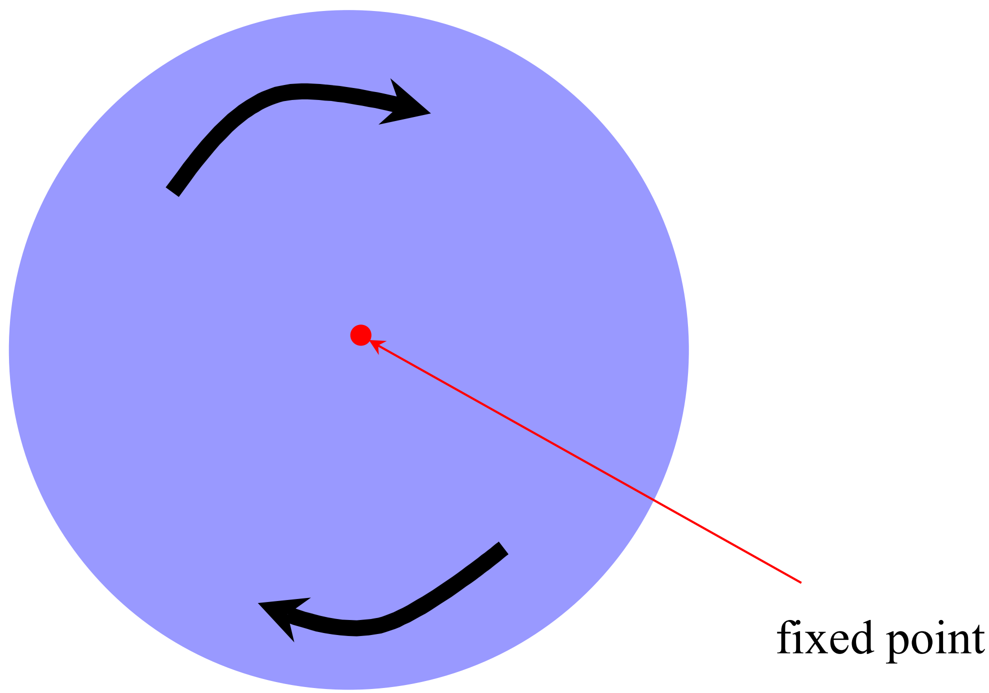
Then there exists an $x \in D$ s.t. $x = f(x)$.

closed and bounded

A few examples, when D is the 2-dimensional disk.



Brouwer's Fixed Point Theorem



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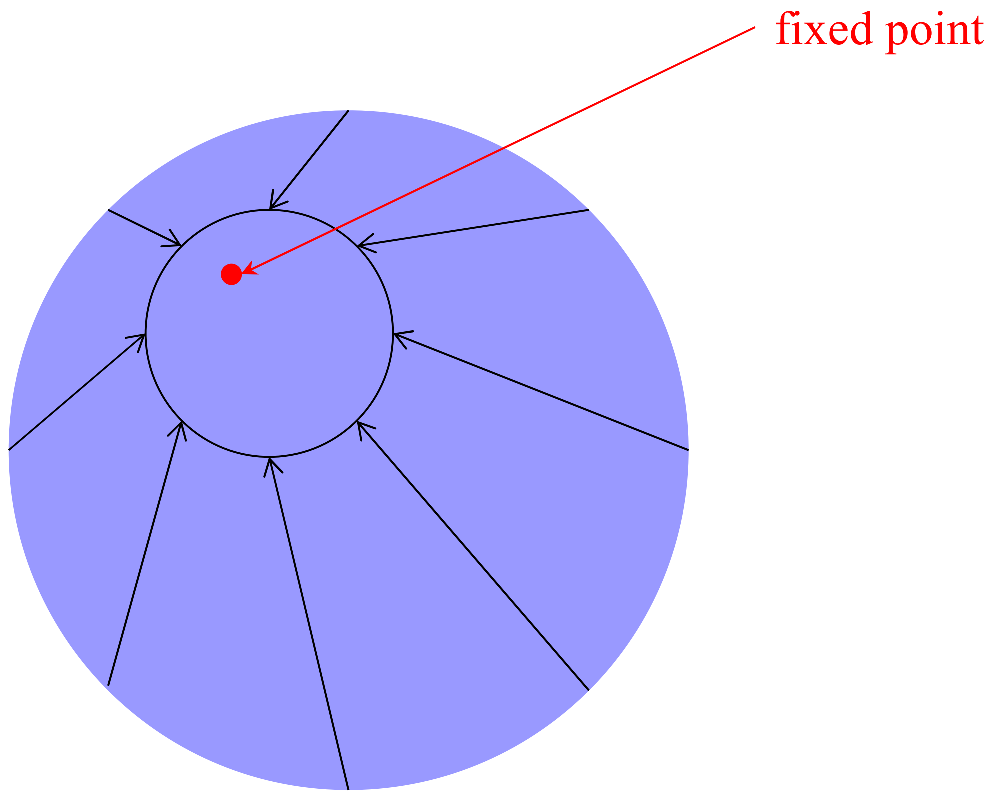

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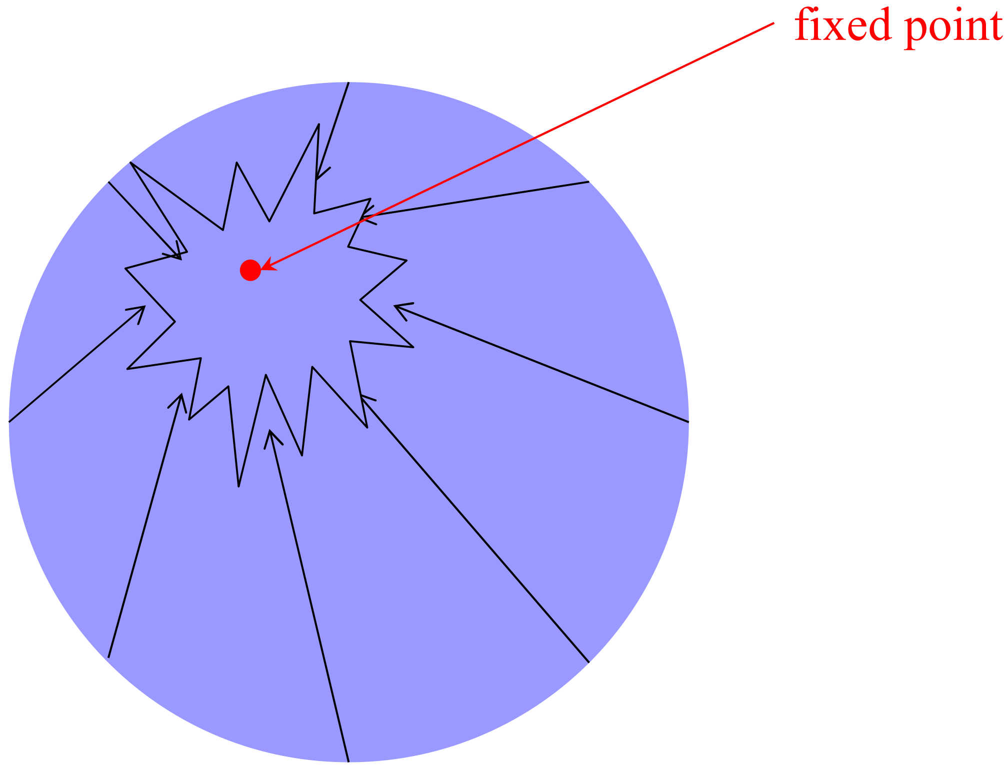


$$Brouwer \Rightarrow Nash$$

(Nash'51)

Visualizing Nash's Proof

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game

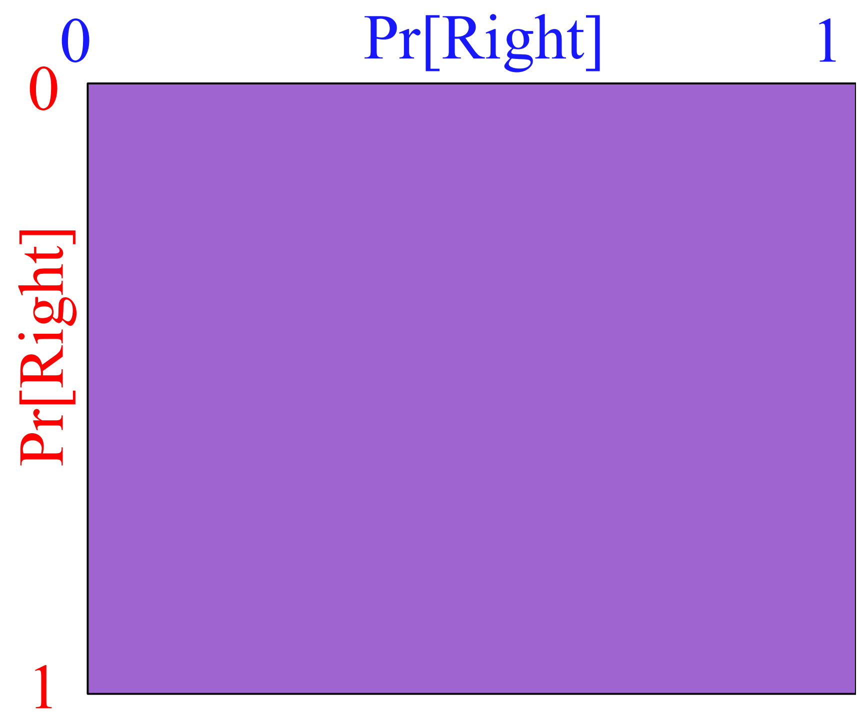
⇨

f: $[0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

Visualizing Nash's Proof

Kick / Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1, -1

Penalty Shot Game



Visualizing Nash's Proof

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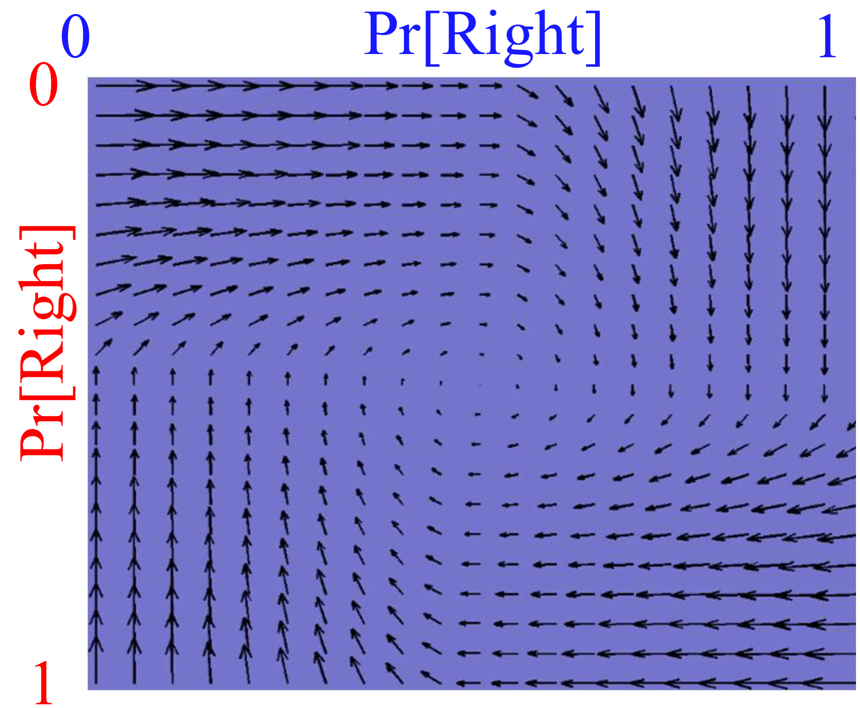

Visualizing Nash's Proof

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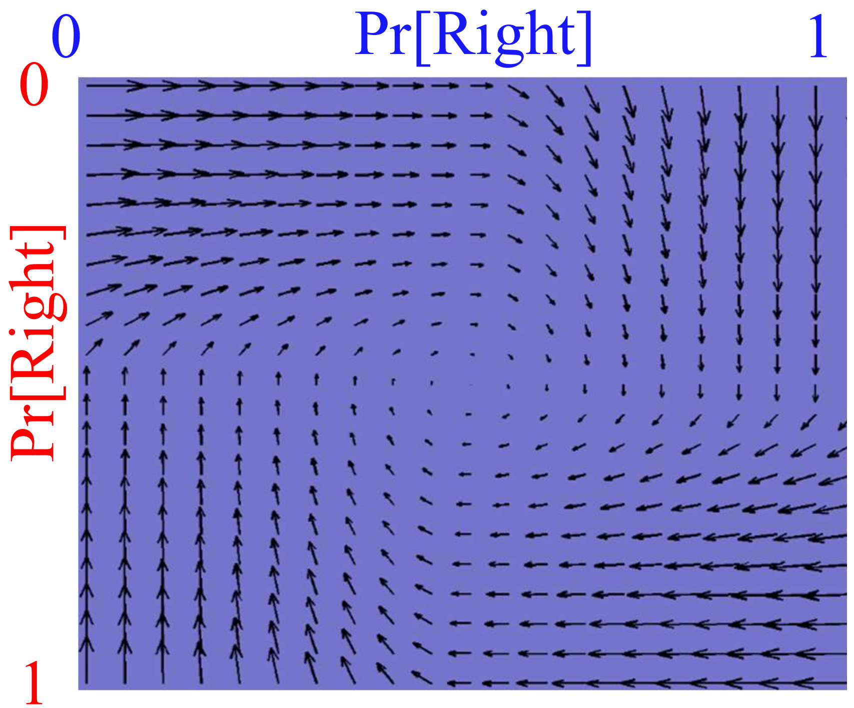
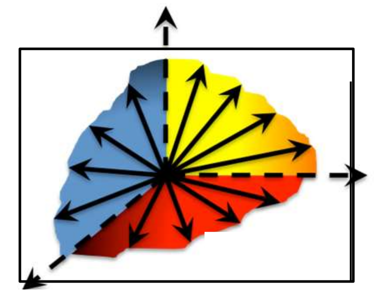
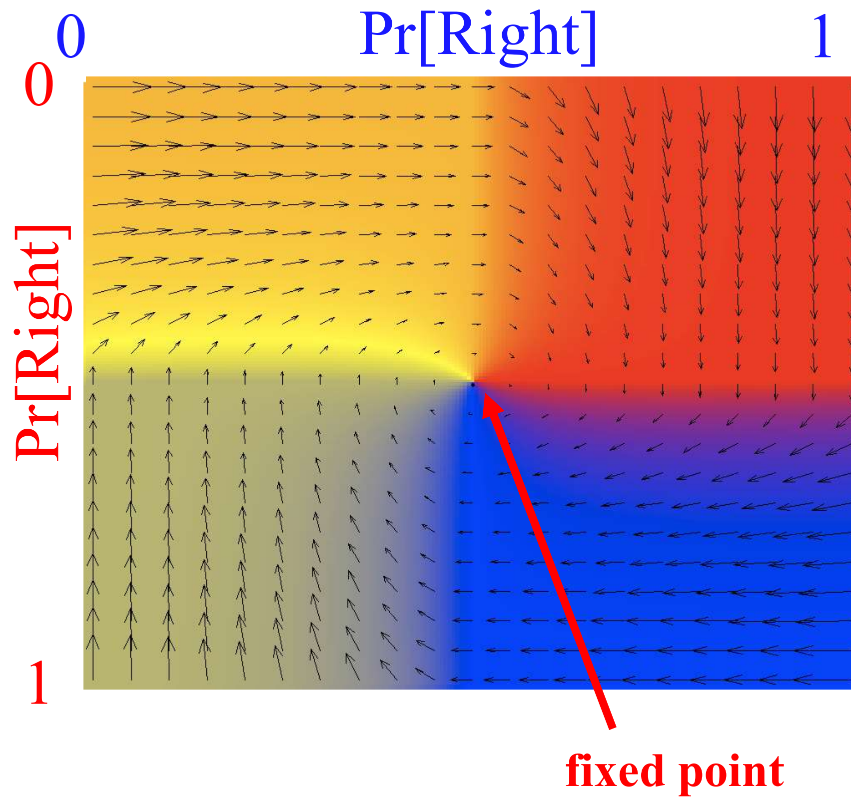


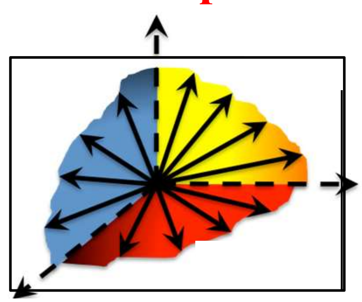


Visualizing Nash's Proof

	½	½
Kick / Dive	Left	Right
½ Left	1 , -1	-1 , 1
½ Right	-1 , 1	1, -1

Penalty Shot Game

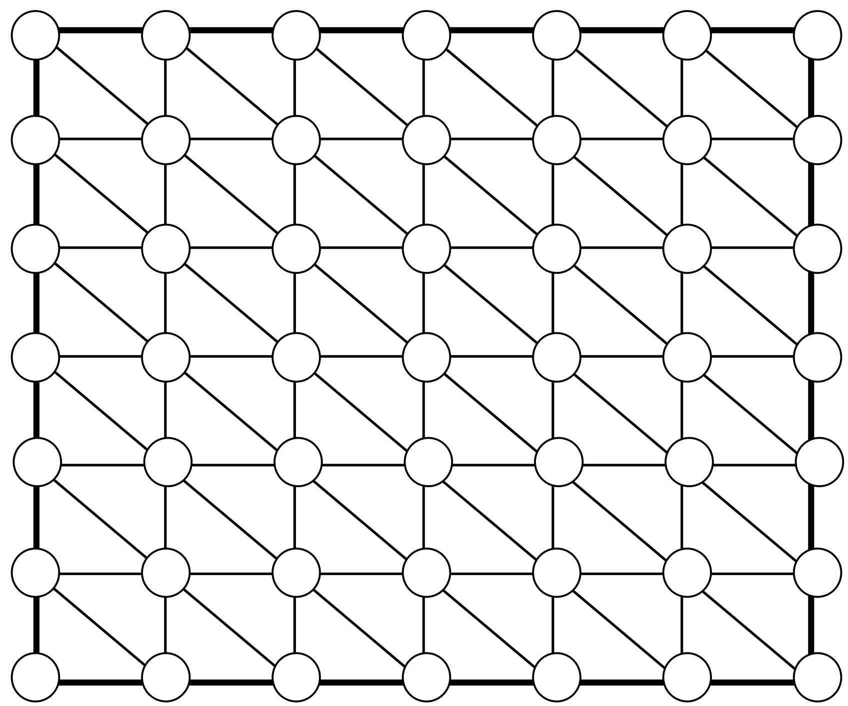


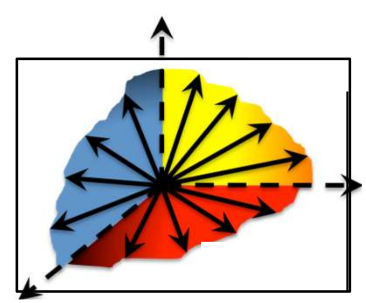


Menu

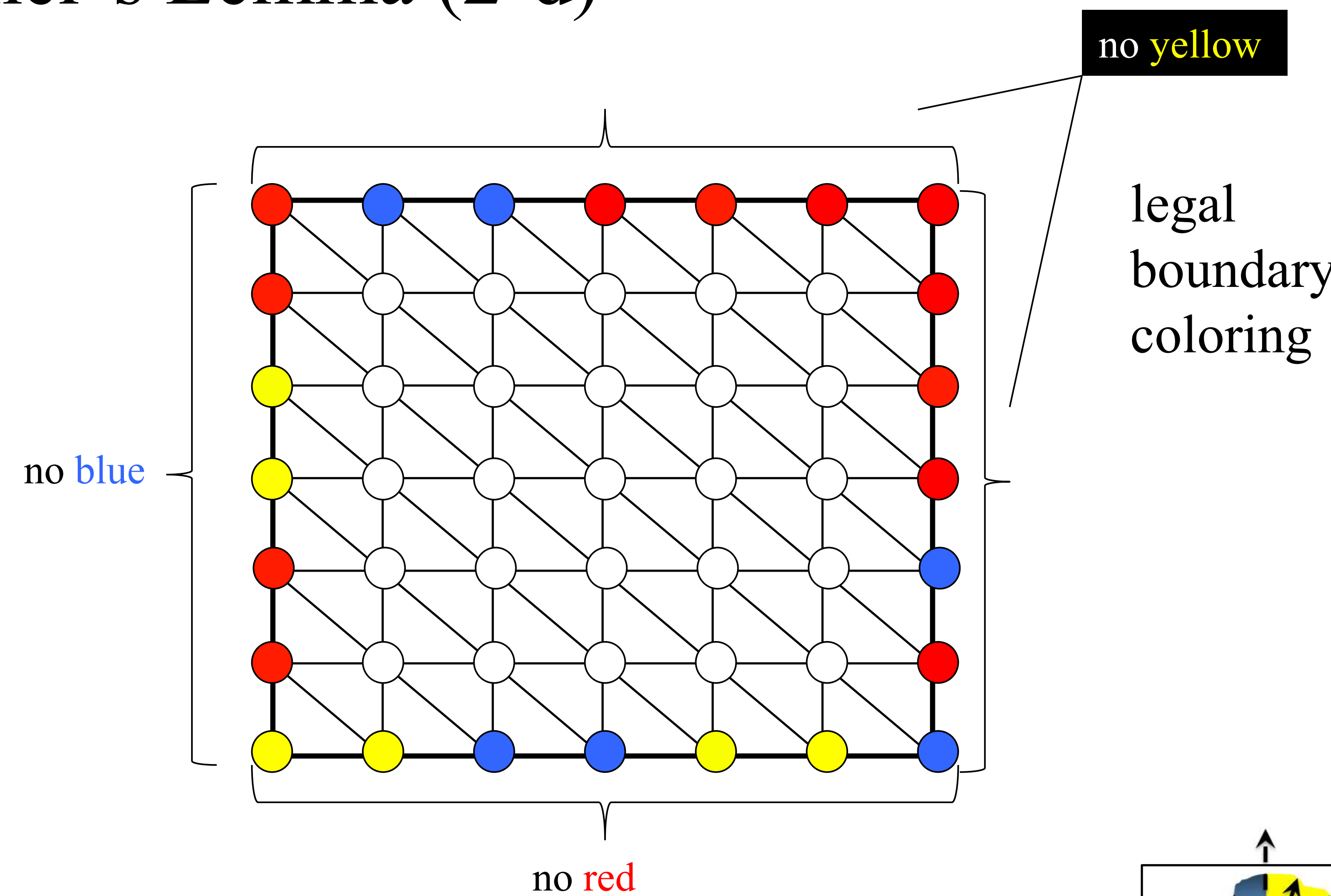
→ Existence Theorems: Nash, Brouwer, **Sperner**

Sperner's Lemma (2-d)

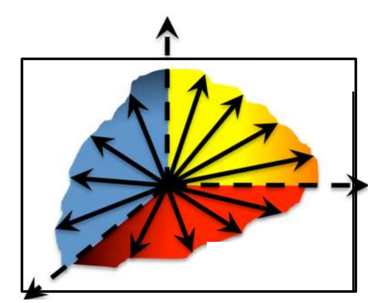




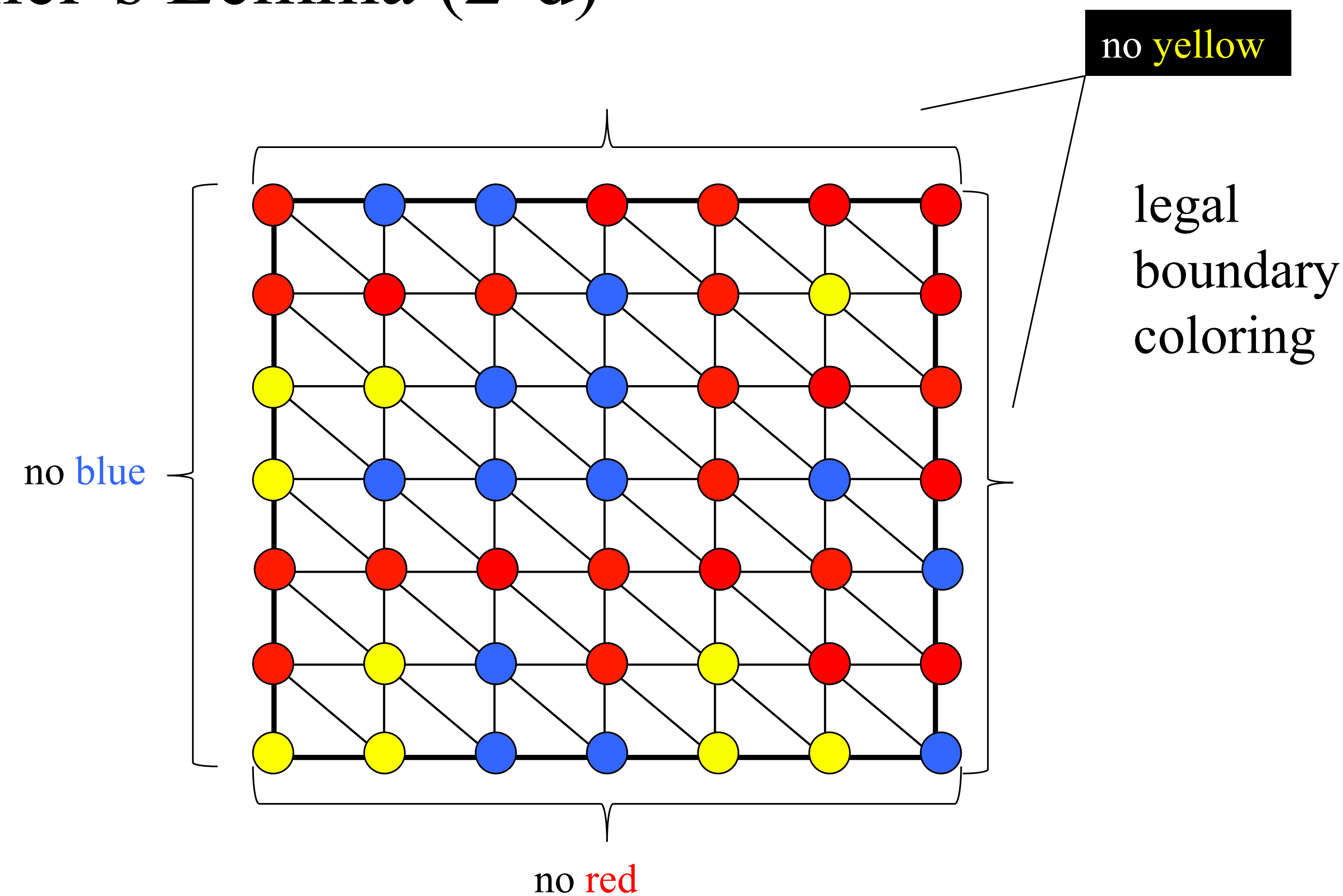
Sperner's Lemma (2-d)



[Sperner 1928]: Color the boundary using three colors in a legal way.



Sperner's Lemma (2-d)



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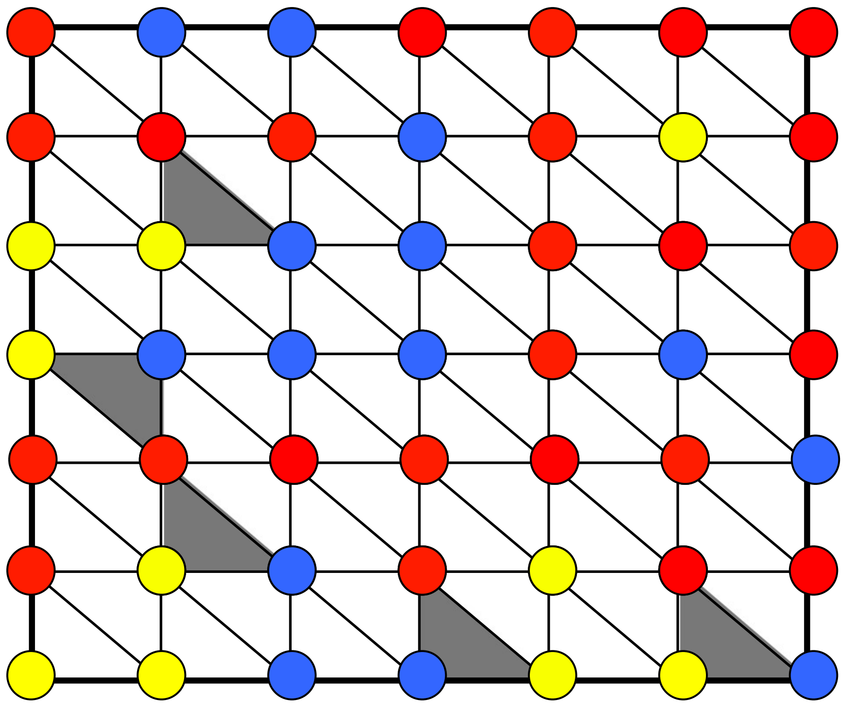
Sperner's Lemma (2-d)



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

$$Sperner \Rightarrow Brouwer$$

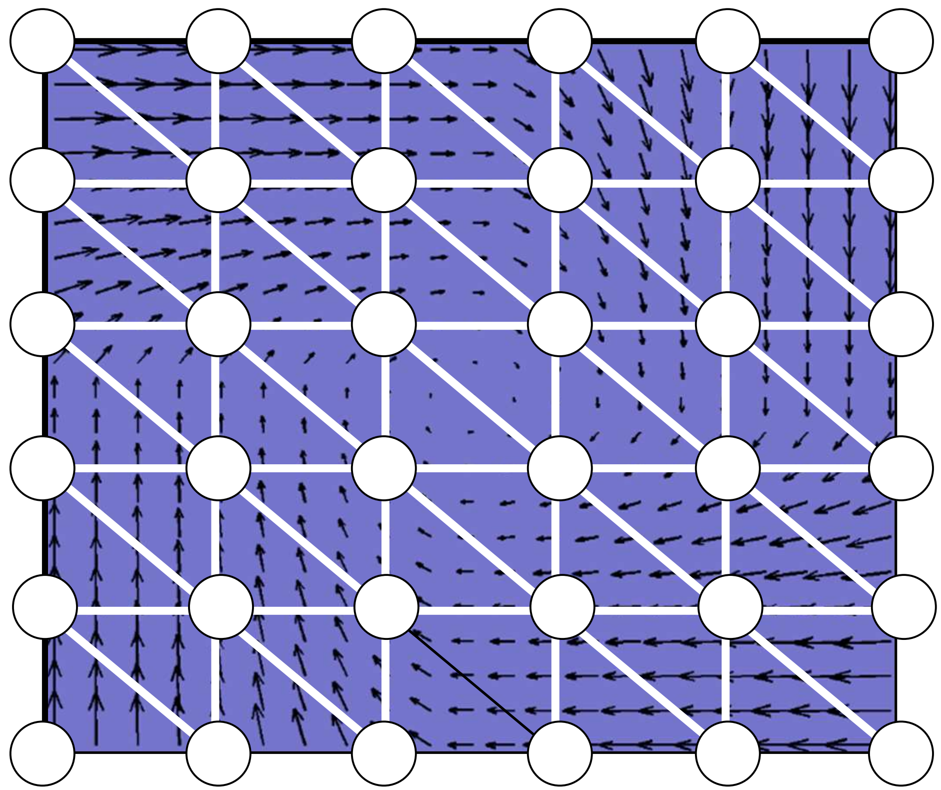
Sperner ⇒ Brouwer (High-Level)

Given $f : [0,1]^2 \rightarrow [0,1]^2$

1) For all $\epsilon > 0$, existence of approximate fixed point $|f(x)\text{-}x| < \epsilon$, can be shown via Sperner's lemma.

2) Then let $\epsilon \rightarrow 0$

For 1): Triangulate $[0,1]^2$;



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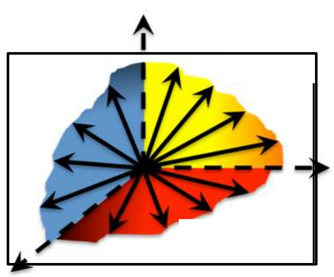
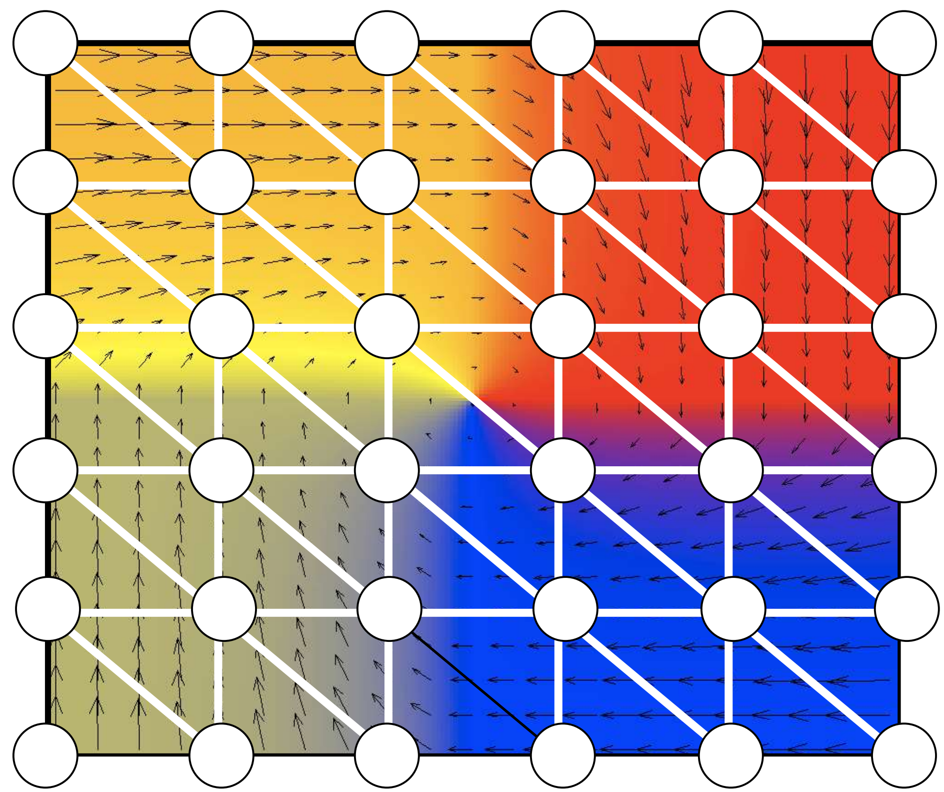
2) Then let $\epsilon \to 0$

For 1): Triangulate $[0,1]^2$;

Color points according to the direction of $(f(x)\text{-}x)$;





Sperner ⇒ Brouwer (High-Level)

Given $f : [0,1]^2 \to [0,1]^2$

1) For all $\epsilon > 0$, existence of approximate fixed point $|f(x)-x| < \epsilon$, can be shown via Sperner's lemma.

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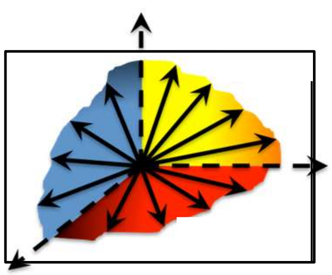

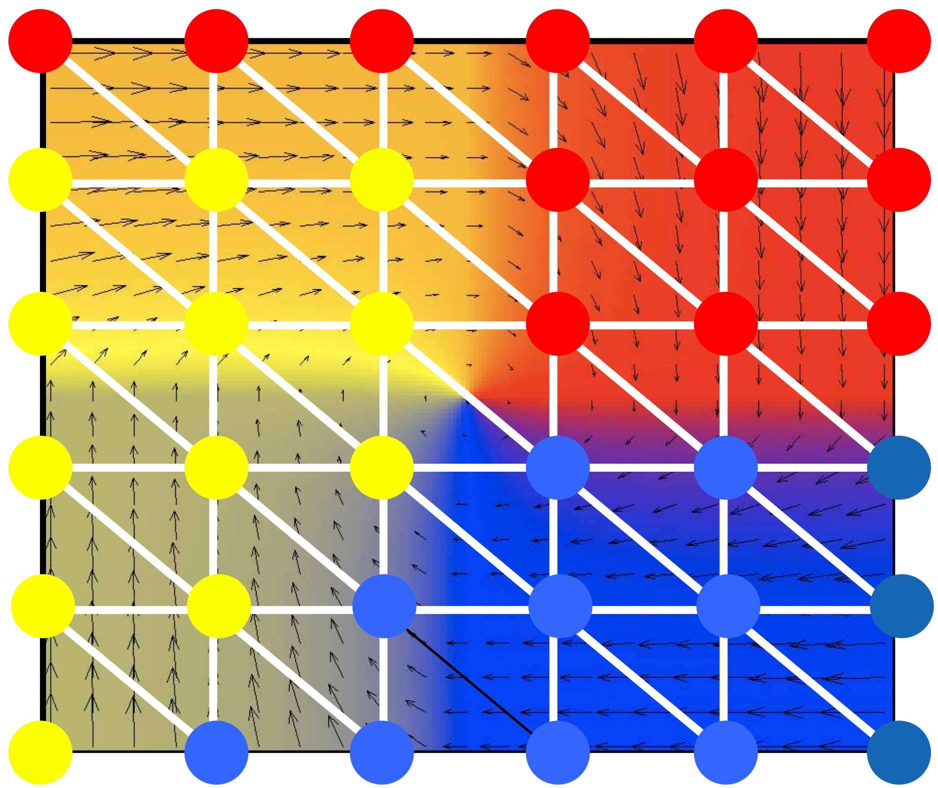

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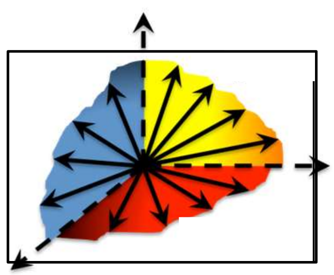
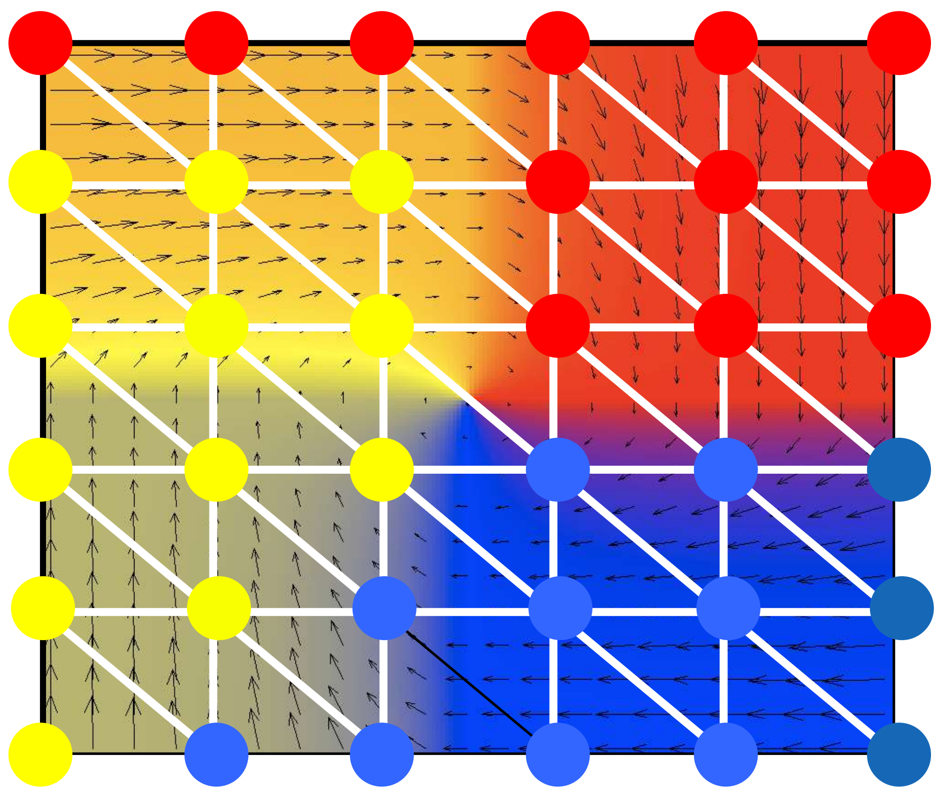
Color points according to the direction of $(f(x)-x)$;

Apply Sperner.





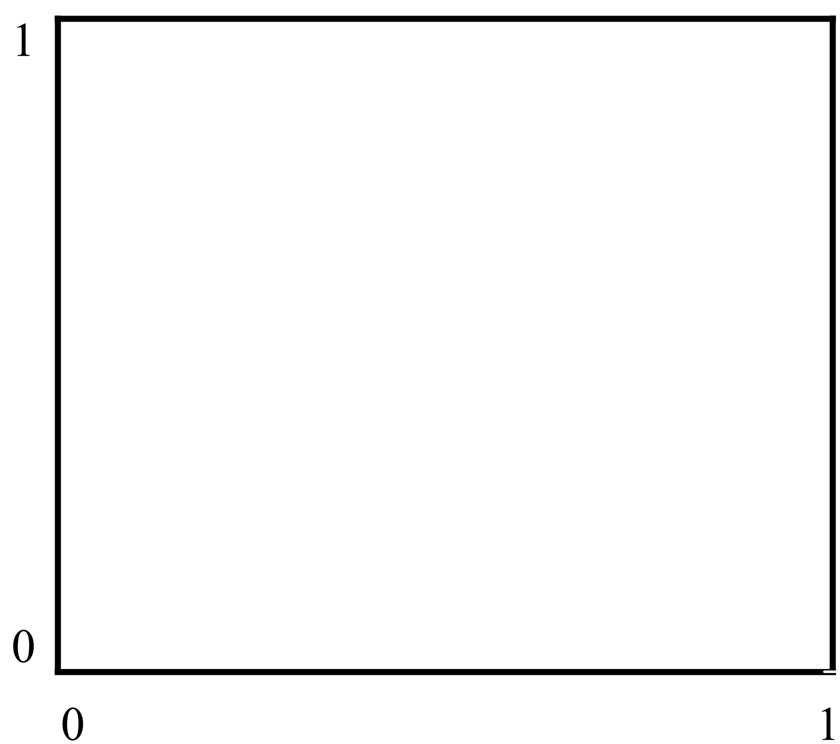
2D-Brouwer on the Square

d be l_∞ norm

Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

$$\forall \epsilon, \exists \delta(\epsilon) > 0, s.t.$$
$$d(x,y) < \delta(\epsilon) \Rightarrow d\big(f(x), f(y)\big) < \epsilon$$

(by the Heine-Cantor theorem)



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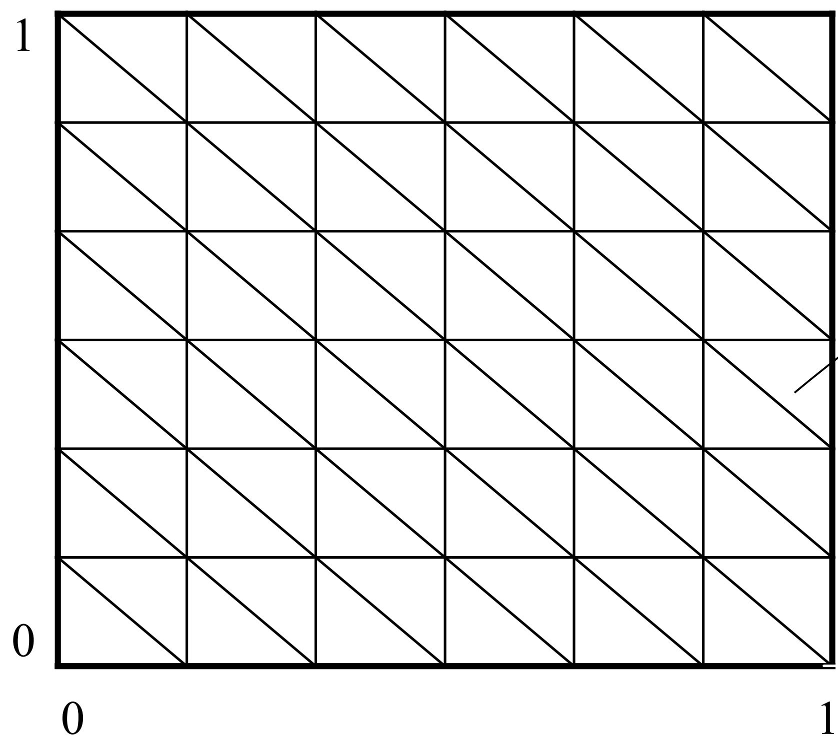

choose some ϵ and triangulate so that the diameter of cells is

$\delta < \delta(\epsilon)$

2D-Brouwer on the Square

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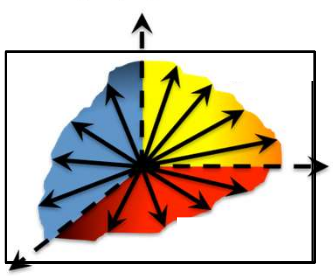
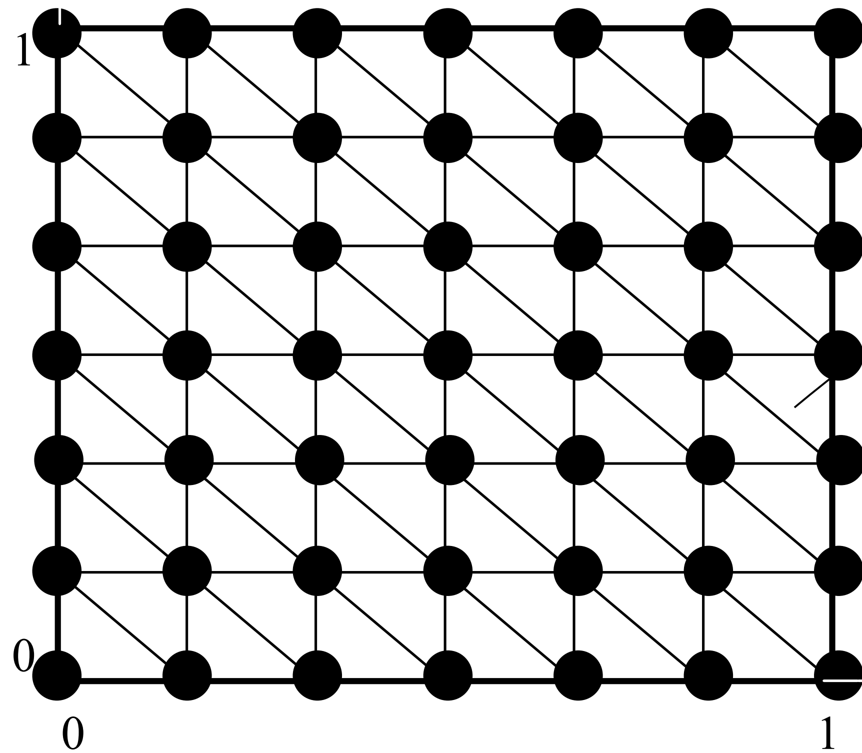
color the nodes of the triangulation according to the direction of

$f(x) - x$





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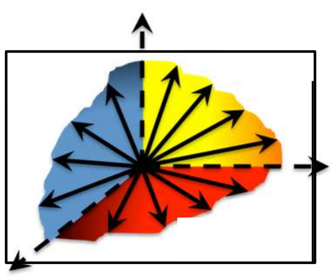
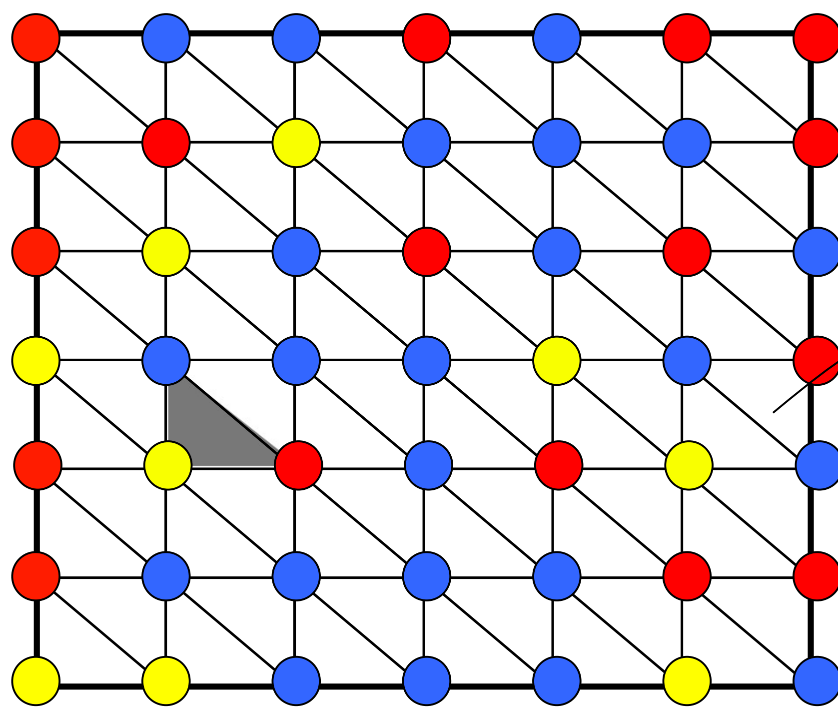


(tie-break at the boundary angles, so that the resulting coloring respects the boundary conditions required by Sperner's lemma)



choose some ϵ and triangulate so that the diameter of cells is

$\delta < \delta(\epsilon)$

find a trichromatic triangle, guaranteed by Sperner

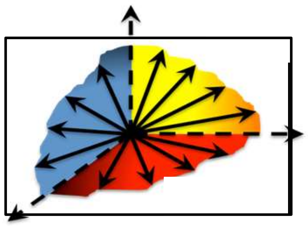
2D-Brouwer on the Square

d be l_∞ norm

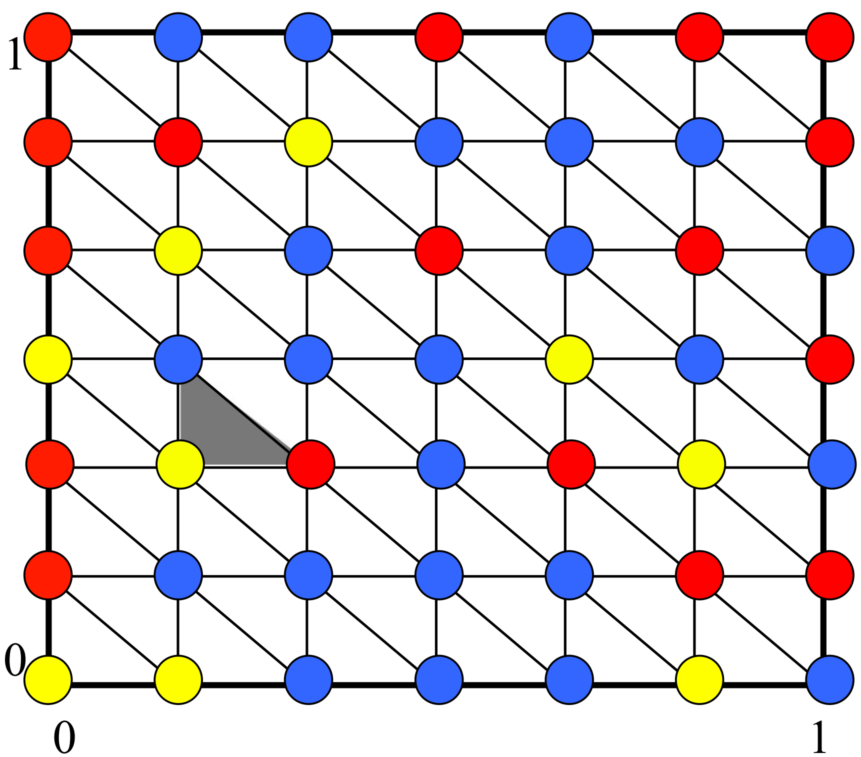
Suppose f: $[0,1]^2 \rightarrow [0,1]^2$, continuous

$\forall \epsilon, \exists \delta(\epsilon) > 0, s.t.$ (by the Heine-Cantor theorem)

$$d(x,y) < \delta(\epsilon) \Rightarrow d\big(f(x), f(y)\big) < \epsilon$$



Choose small enough grid size so that..



Claim: If z a corner of a trichromatic triangle, then

Choosing $\delta = \min\{\delta(\epsilon), \epsilon\}$

$$|f(z) - z|_\infty < c\delta, \qquad c > 0$$

2D-Brouwer on the Square

Finishing the proof of Brouwer's Theorem (Compactness):

- pick a sequence of epsilons: $\epsilon_i = 2^{-\mathrm{i}}$, $\mathrm{i} = 1, 2, \ldots$
- define a sequence of triangulations of diameter: $\delta_i = \min\{\delta(\epsilon_i), \epsilon_i\}$, $i = 1,2, \ldots$
- pick a trichromatic triangle in each triangulation. Its corner be z_i, $i = 1,2, \ldots$
- by compactness, this sequence has a converging subsequence w_i, $i = 1,2, \ldots$ with limit point w^*

Claim: $f(w^*) = w^*$

Proof: Define the function $g(x) = d(f(x), x)$. Clearly, g is continuous since $d(.,.)$ is continuous and so is f. It follows from continuity that

$$g(w_i) \to g(w^*), as\ i \to \infty$$

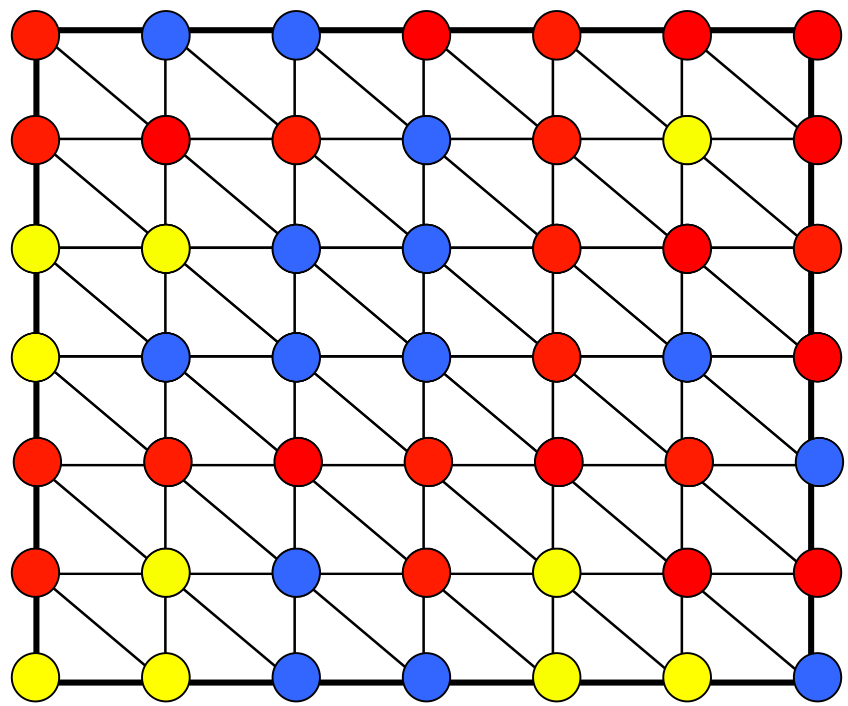
But $0 \leq g\left(w^i\right) \leq c2^{-i}$. Hence, $g(w_i) \to 0$. It follows that $g(w^*) = 0$.

Therefore, $d(f(w^*), w^*) = 0 \Rightarrow f(w^*) = w^*$ ∎

Menu

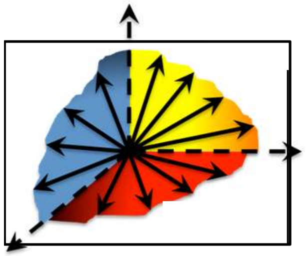
- Existence Theorems: Nash, Brouwer, Sperner
- (Constructive) proof of Sperner $\rightarrow$ PPAD.

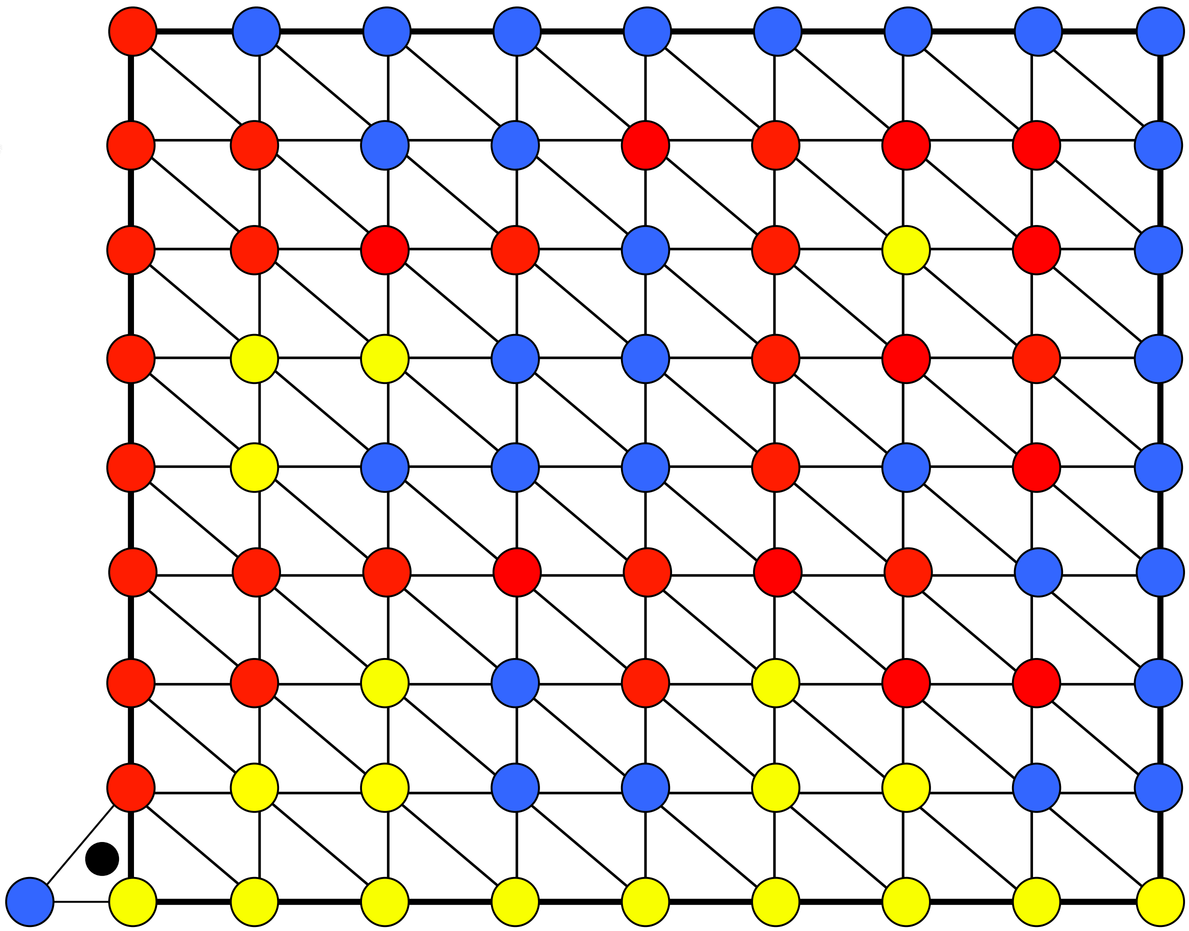
Proof of Sperner's Lemma



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma





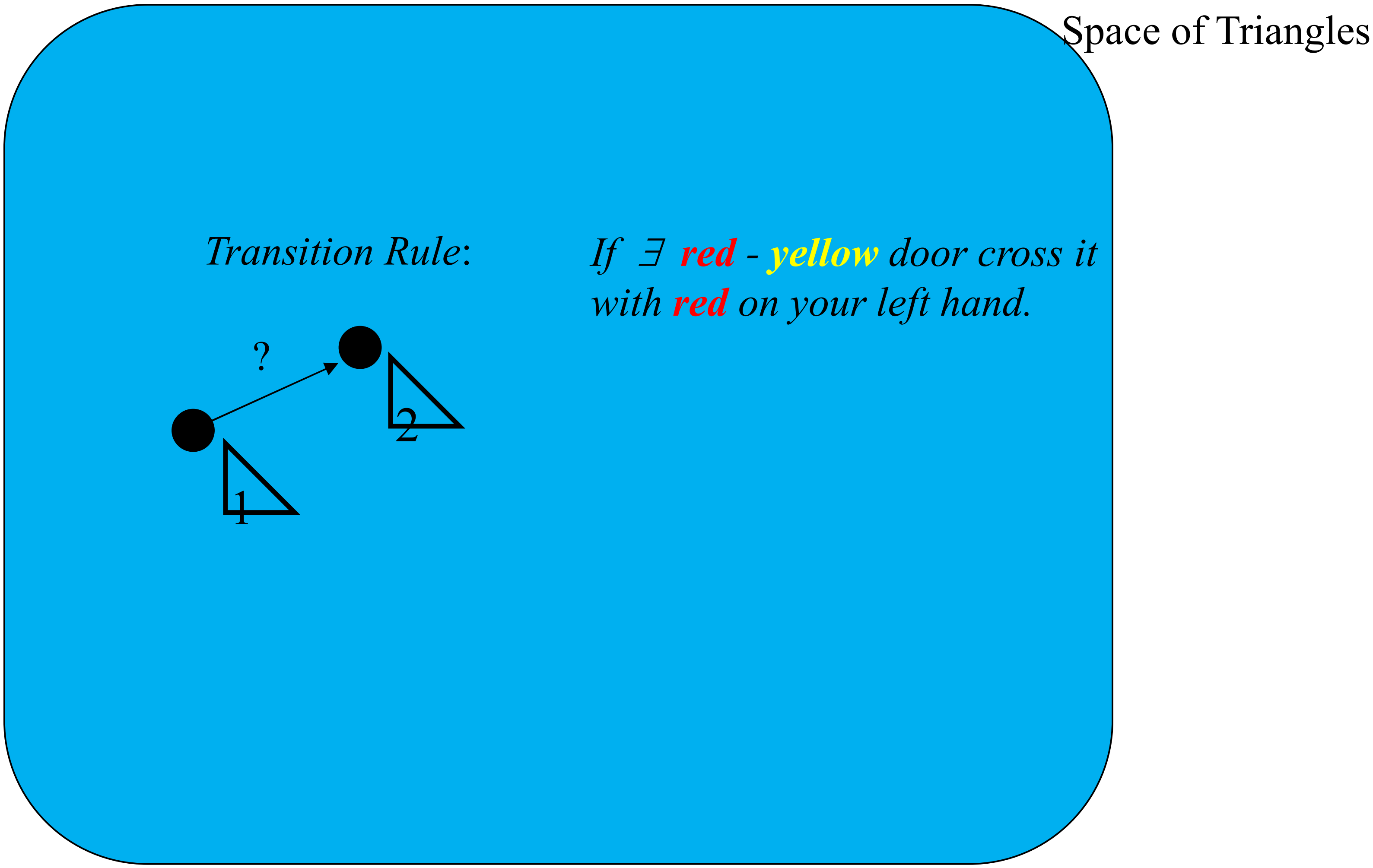
For convenience introduce an outer boundary, that does not create new tri-chromatic triangles.

Also introduce an artificial tri-chromatic triangle.

Next define a directed walk starting from the artificial tri-chromatic triangle.

[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner’s Lemma



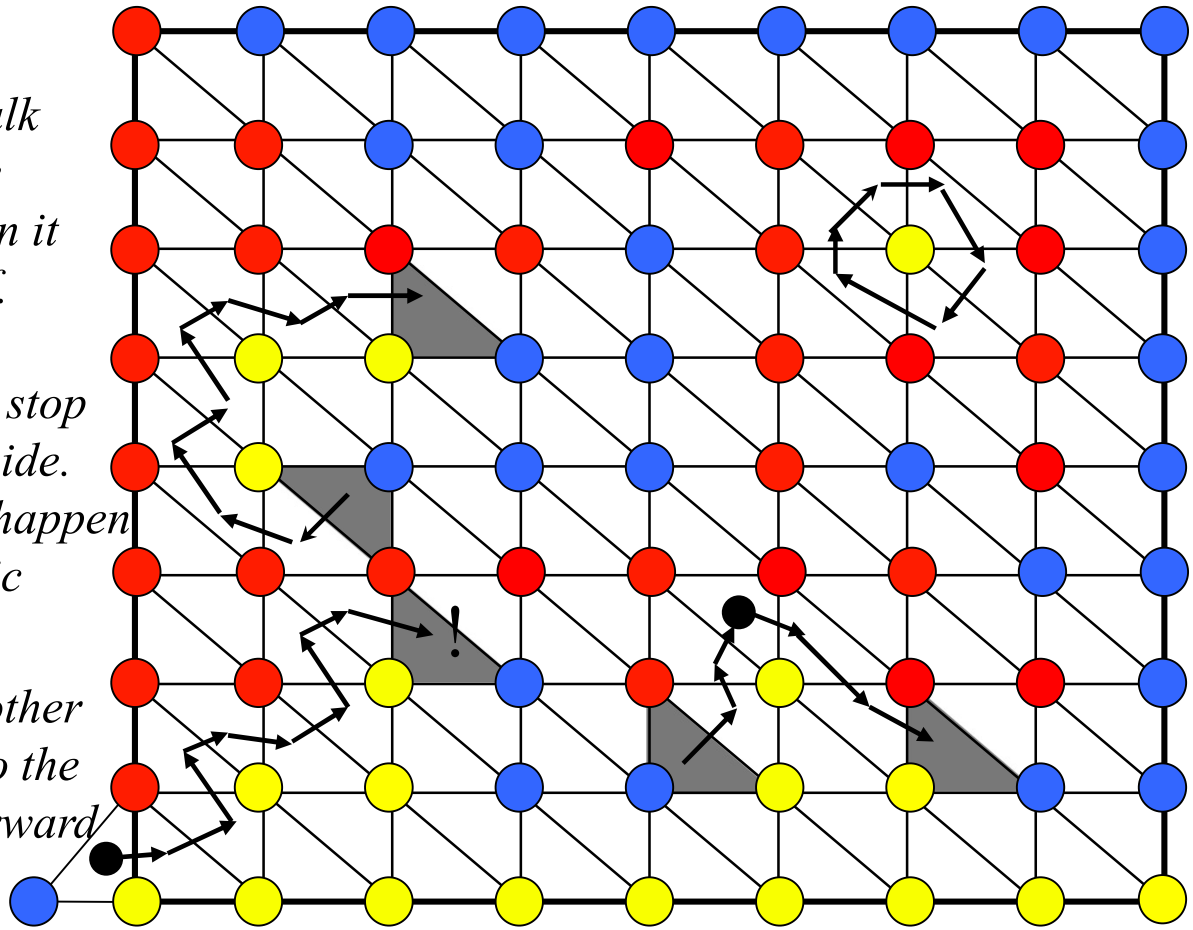
[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner’s Lemma

Claim**:** *The walk cannot exit the square, nor can it loop into itself.*

Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle…

Starting from other triangles we do the same going forward or backward.



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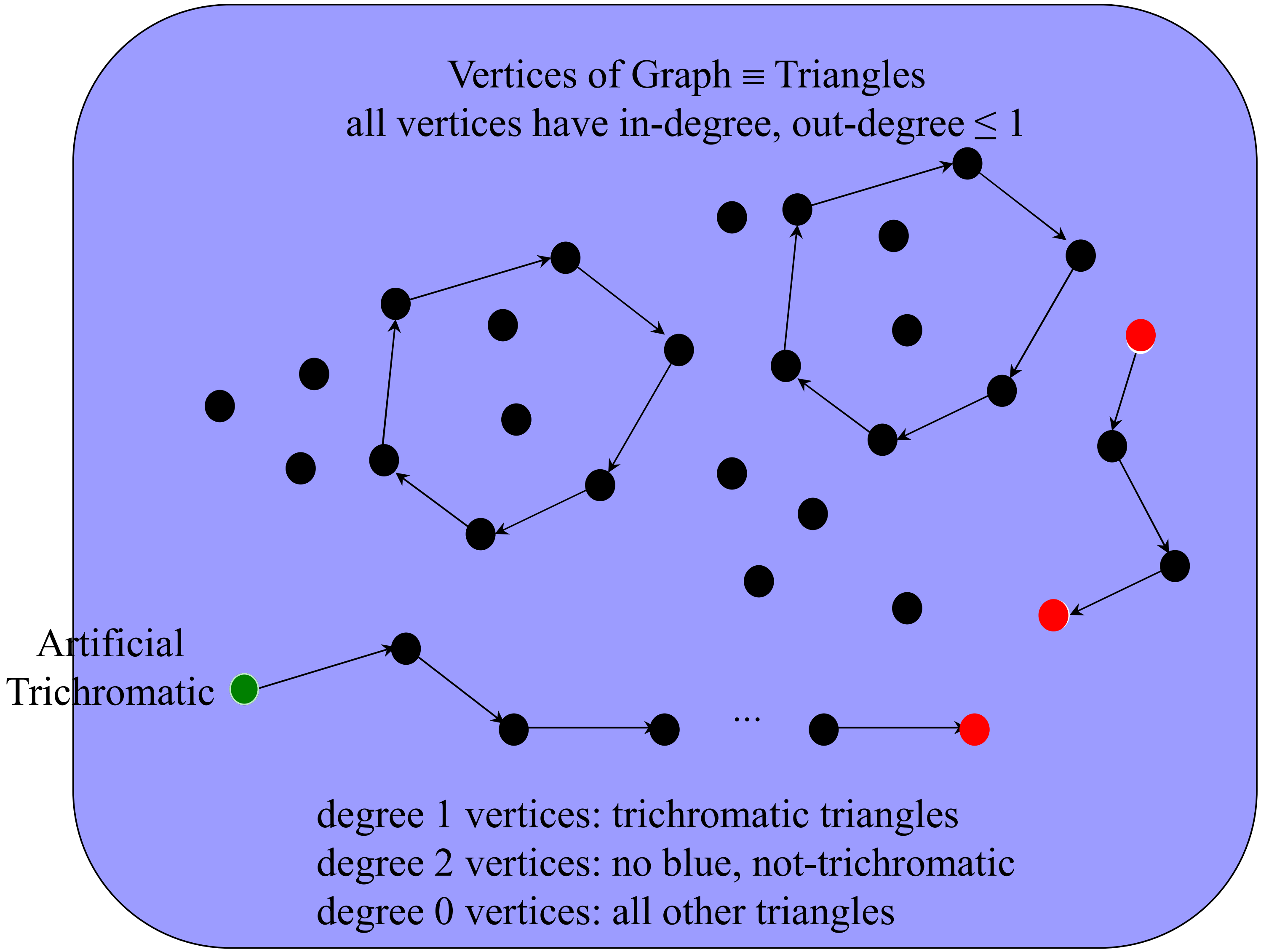
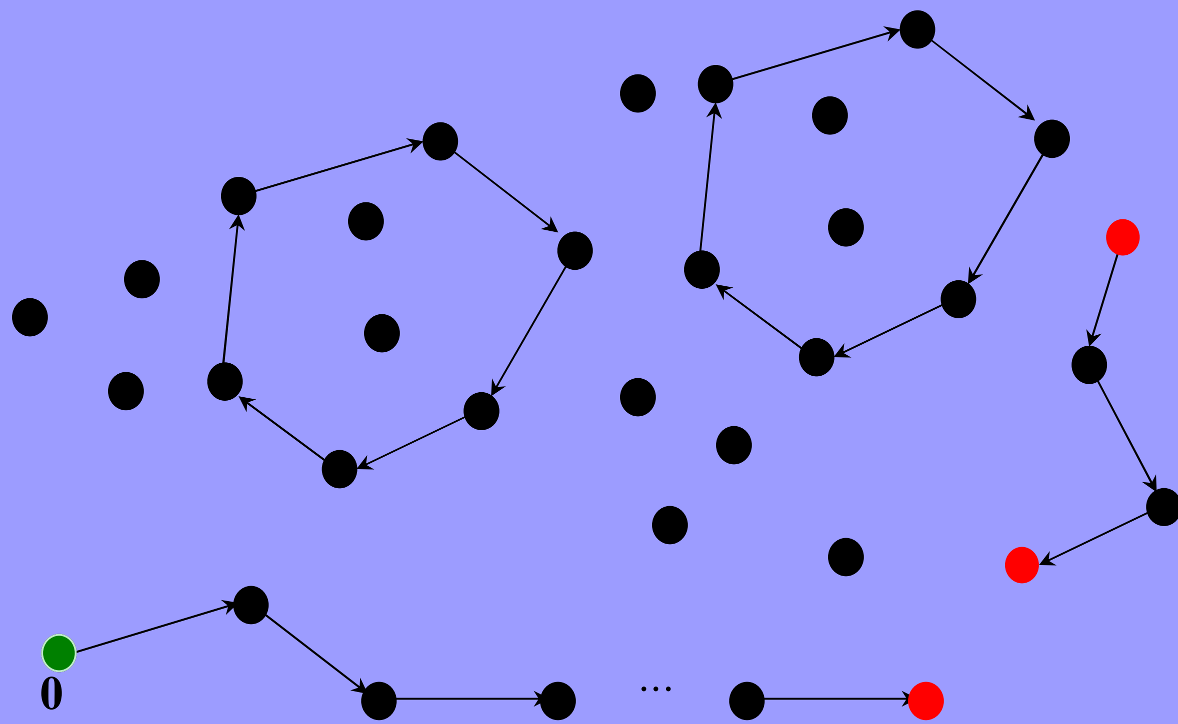
Proof Structure: A directed parity argument



Proof: ∃ at least one trichromatic (artificial one) ➔ ∃ another trichromatic

Recall: Lemke-Howson Structure for 2-Nash

Vertices of Graph ≡ vertices of the polyhedron.



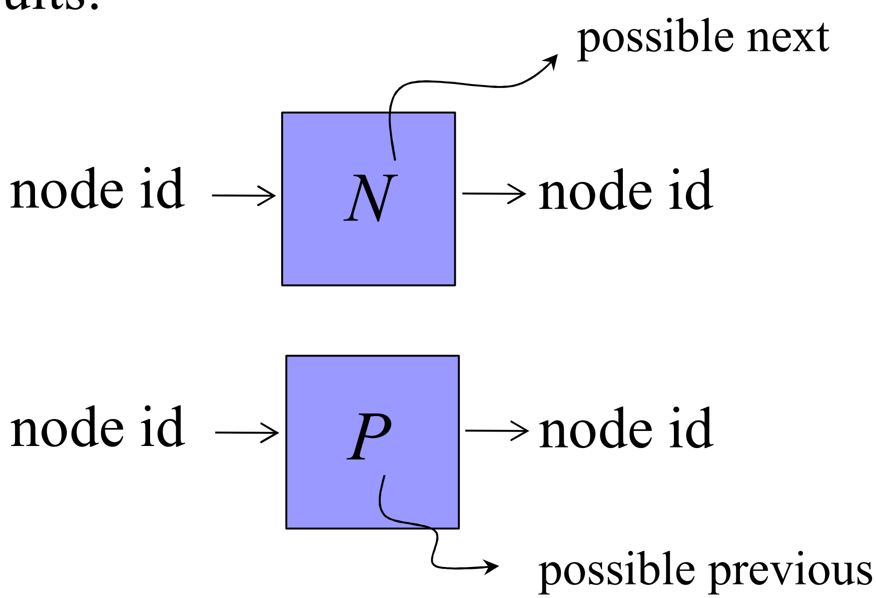
degree 1 vertices: fully-labeled
degree 2 vertices: no label 1, 1-almost labeled
degree 0 vertices: all other vertices of the polytope

Proof: 0 fully-labeled ➔ ∃ another fully-labeled

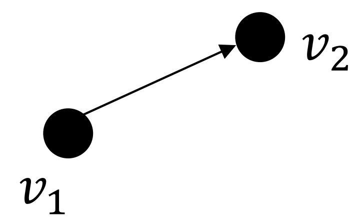
The PPAD Class [Papadimitriou ’94]

(Polynomial Parity Argument for Directed Graph)

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:



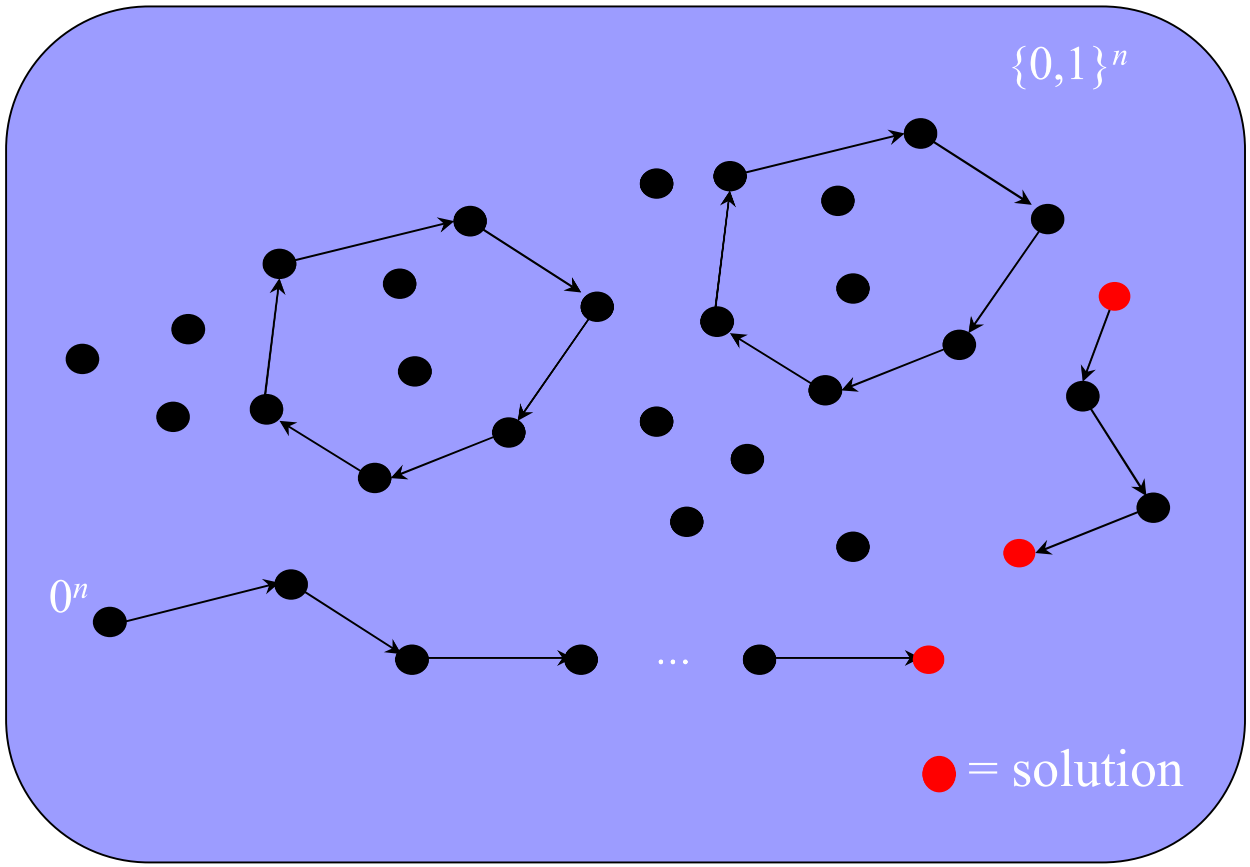
$N(v_1) = v_2 \;\&\; P(v_2) = v_1$

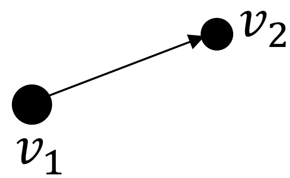


END OF THE LINE: P and N are given. If 0^n is an unbalanced node, find another unbalanced node. Otherwise output 0^n.

PPAD = { *Problems reducible to* END OF A LINE }

END OF A LINE



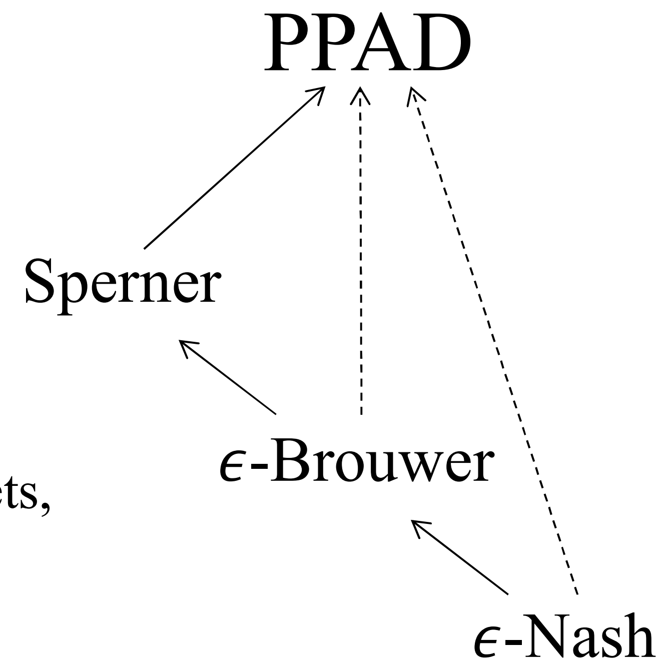


$$\Leftrightarrow P(v_2) = v_1 \;\&\; N(v_1) = v_2$$

[Papadimitriou '94]

PPAD-complete:

Nash eq. (even 2-player games),
Market eq., Sperner, Brouwer,
win-lose games, sparse games,
competitive eq. with equal income,
clearing payments in financial markets,
Fractional hypergraph matching,
Fractional stable path problem, …



ϵ-Brower: Given $f: D \to D$, find $x \in D, s.t. |f(x) - x| < \epsilon$

ϵ-Nash: Profile from which no player can deviate and gains by more than ϵ.

∵Exact could be irrational

Menu

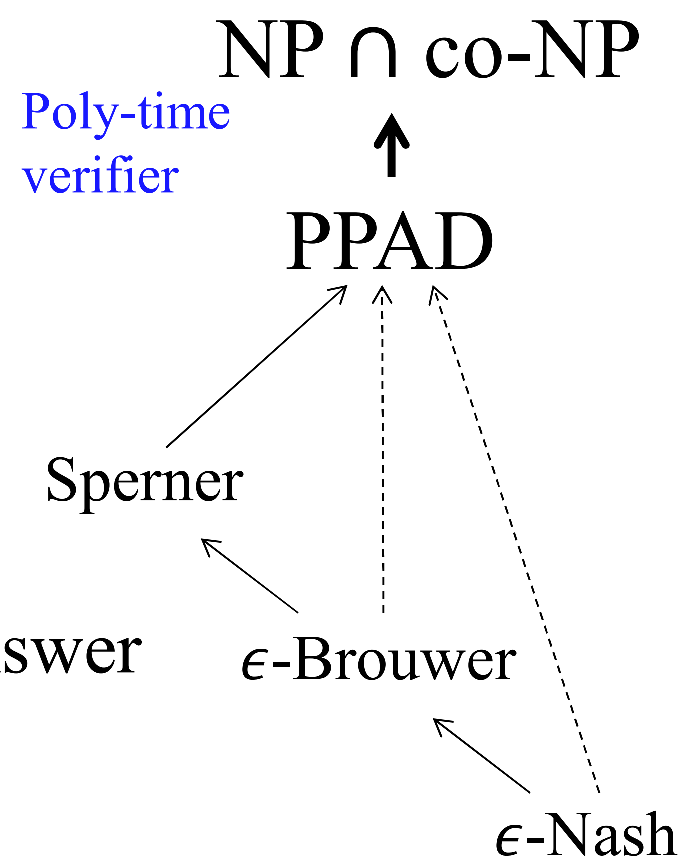
- Existence Theorems: Nash, Brouwer, Sperner
- (Constructive) proof of Sperner, and PPAD
- Why not use NP?
  Total Search problems.

NP, co-NP vs PPAD

Can they be NP-hard? **NO!**

If there exists a solution?

- NP: poly-time verifier for YES answer
- co-NP: poly-time verifier for NO answer



- Here the answer is always YES!
  - The problem is to find a solution.

Function NP (FNP): Search problems

Either find a solution or say there is none!

Problem $L \in$ FNP has a poly-time verifier A_L s.t. $A_L(x,y) = 1$ if y is a soln. of $x \in L$

It is ***poly-time (Karp) reducible*** to another problem L' ∈ FNP, associated with $A_{L'}$, iff there exist poly-time functions *f, g* such that

(i) $f: \{0,1\}^* \to \{0,1\}^*$ maps inputs x to L into inputs $f(x)$ to L'

(ii) $\forall$ x,y: $A_{L'}(f(x), y)=1 \Rightarrow A_L(x, g(y))=1$
$\forall$ x: $A_{L'}(f(x), y)=0, \forall y \Rightarrow A_L(x, y)=0, \forall y$

can't reduce SAT to SPERNER, NASH or BROUWER

A search problem L' is ***FNP-complete*** iff

$L' \in$ FNP

$\forall L \in$ FNP, L is poly-time reducible to L'.

e.g. SAT

SPERNER, NASH, BROUWER ∈ FNP.

Total Function NP

Function NP (FNP): Search problems

Either find a solution or say there is none!

Total FNP: A search problem is called ***total*** iff a solution is guaranteed

$$\text{PPAD} \subseteq \text{TFNP} \subseteq (\text{NP} \cap \text{co-NP})$$

Complexity Theory of TFNP:

1. identify the combinatorial argument of existence, responsible for making these problems total;

2. define a complexity class inspired by the argument of existence;

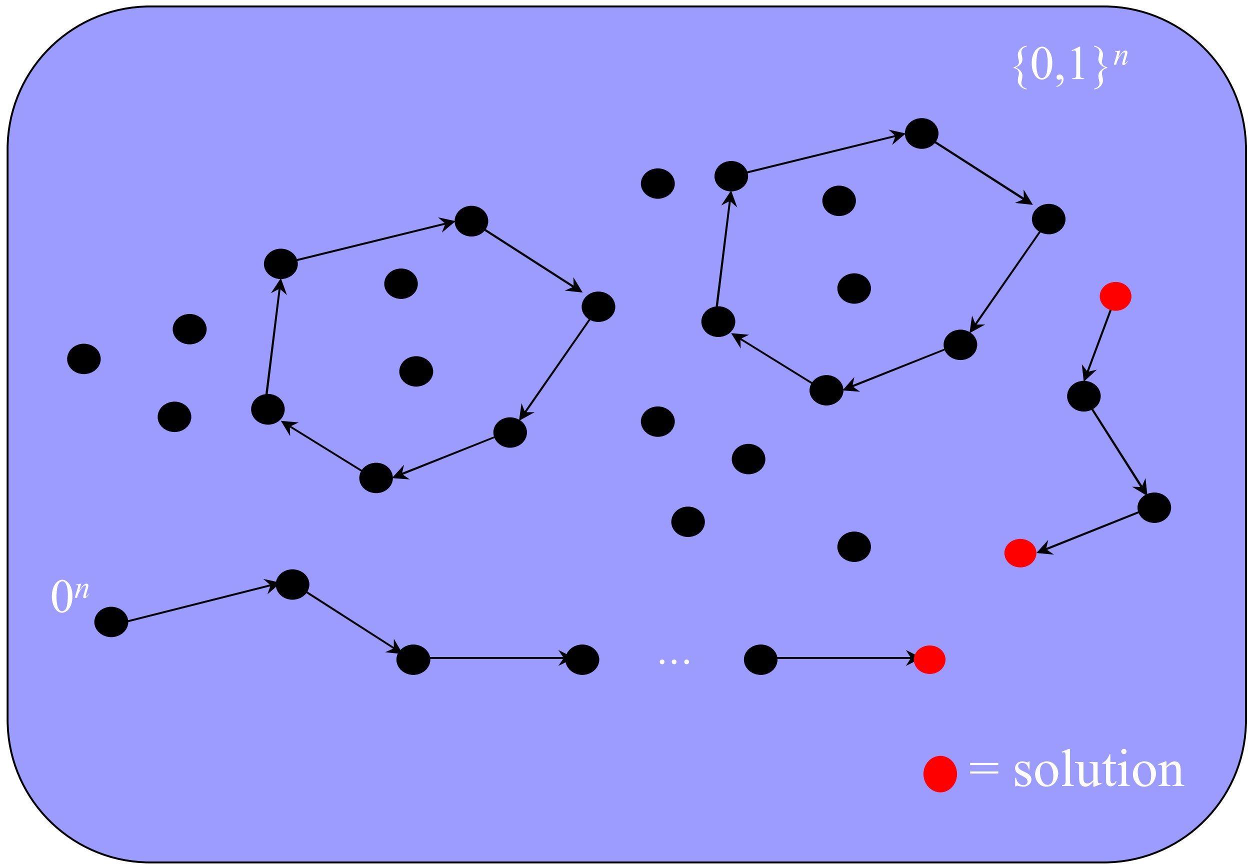
3. make sure that the complexity of the problem was captured as tightly as possible (via completeness results).

PPAD:

In & out degree$\leq$ 1

0^n with in+out=1

∃ another such node



Other arguments of existence, and resulting complexity classes

"If an undirected graph has a node of odd degree, then it must have another."

PPA

"Every directed acyclic graph must have a sink."

PLS

"If a function maps n elements to n-1 elements, then there is a collision."

PPP

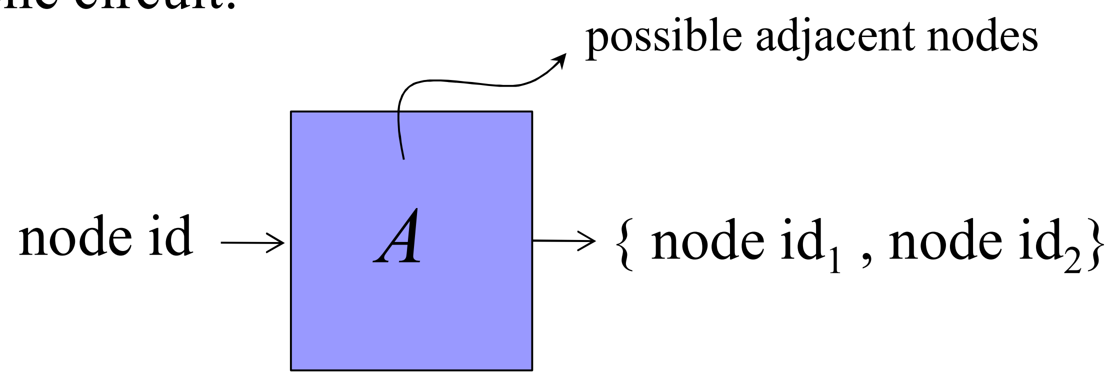
Formally?

PPA: Polynomial Parity Argument

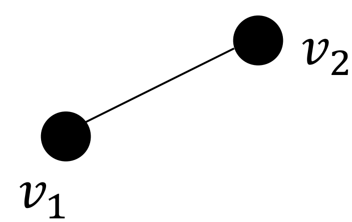
[Papadimitriou '94]

"If a graph has a node of odd degree, then it must have another."

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



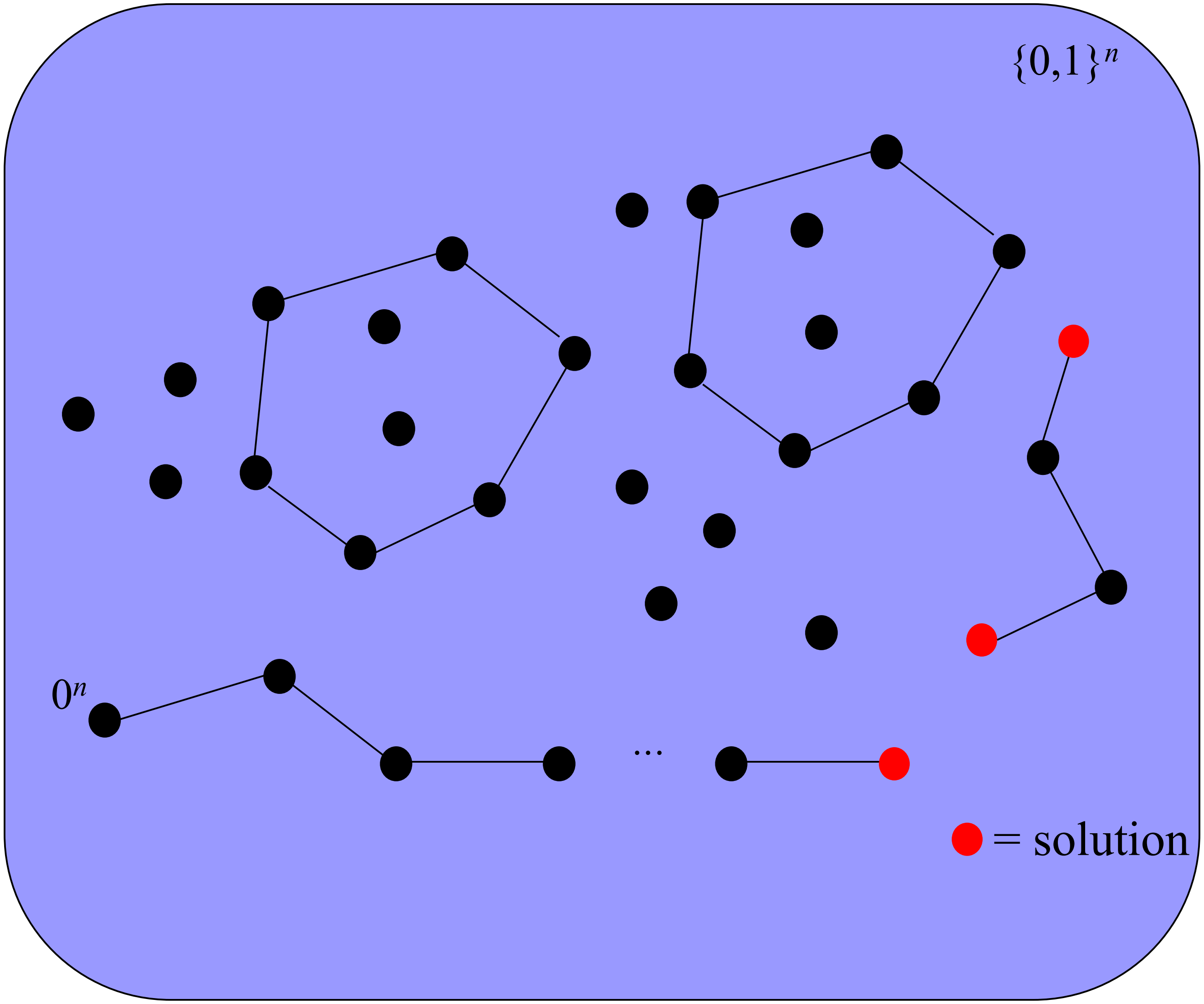
$v_1 \in A(v_2)$ & $v_2 \in A(v_1)$



ODD DEGREE NODE: Given C, if 0^n has odd degree, find another node with odd degree. Otherwise say "yes".

PPA = { *Search problems in FNP reducible to* ODD DEGREE NODE}

The Undirected Graph

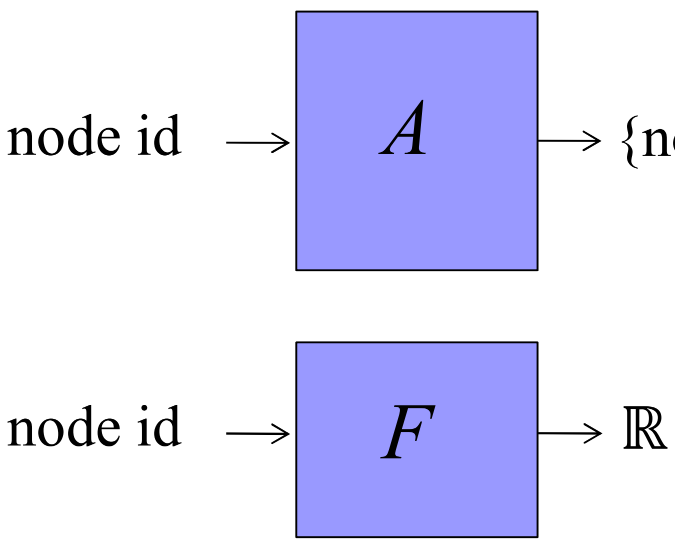


PLS: Polynomial Local Search

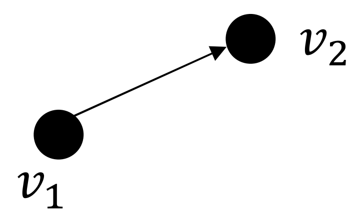
[Johnson, Papadimitriou, Yannakakis '89]

"Every DAG has a sink."

Suppose that a DAG with vertex set $\{0,1\}^n$ is defined by two circuits:



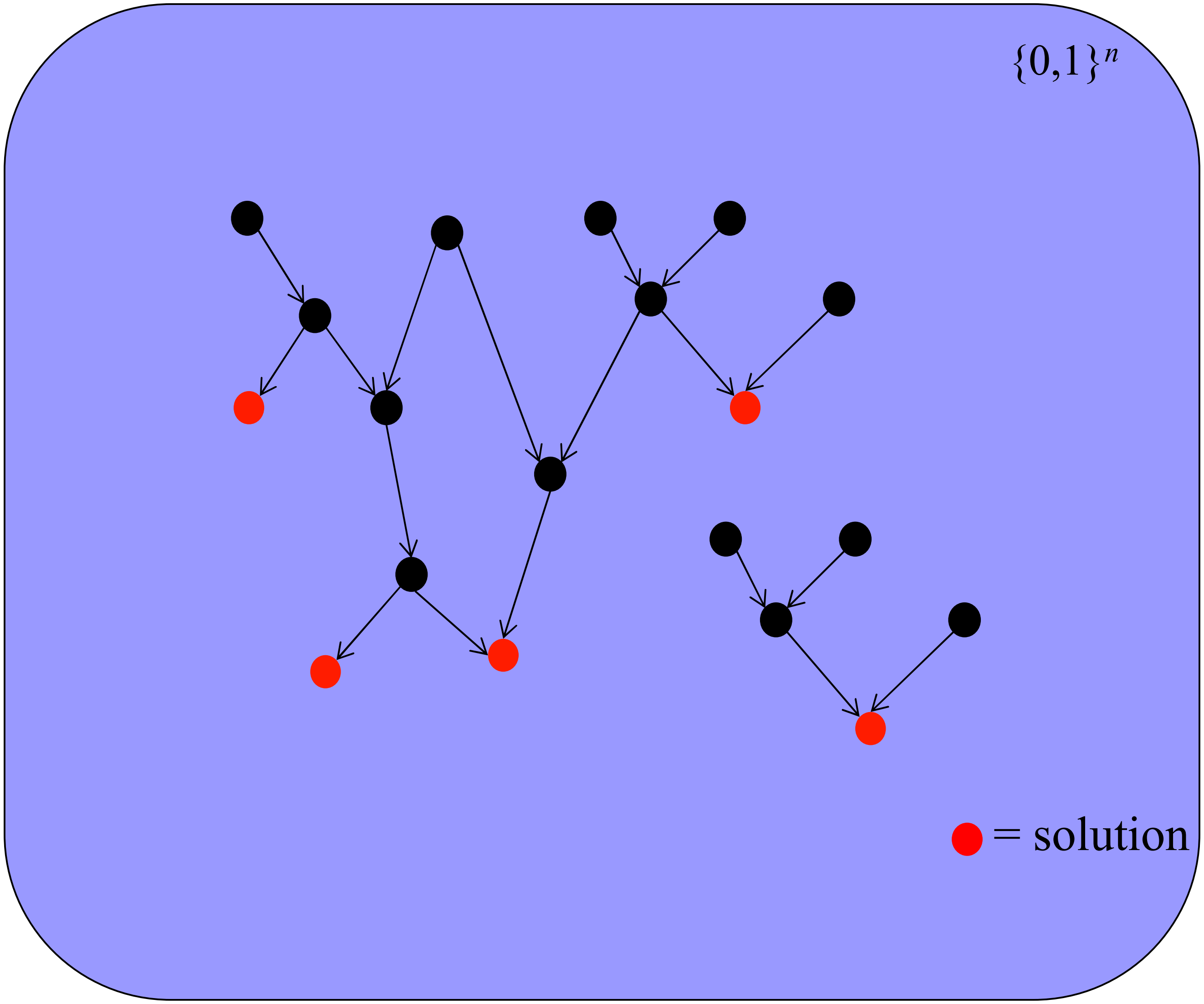
$v_2 \in A(v_1)$ & $F(v_2) > F(v_1)$



FIND SINK. Given *C, F*: Find *x* s.t. $F(x) \geq F(y)$, for all $y \in C(x)$.

PLS = { *Search problems in FNP reducible to* FIND SINK}

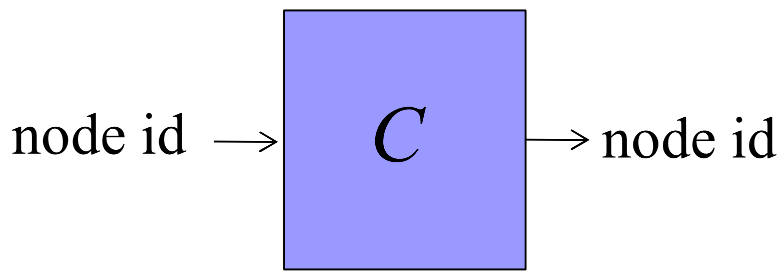
The DAG



PPP: Polynomial Pigeonhole Principle
[Papadimitriou '94]

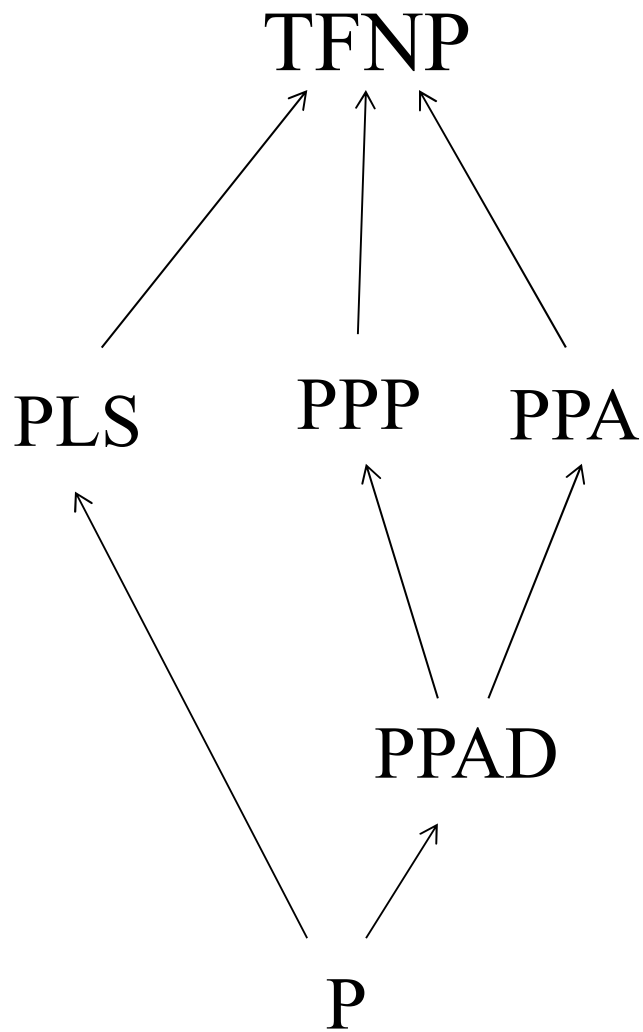
"If a function maps n elements to n-1 elements, then there is a collision."

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



COLLISION. Given C: Find x s.t. $C(x) = 0^n$; or find $x \neq y$ s.t. $C(x)=C(y)$.

PPP = $\{$ *Search problems in FNP reducible to* COLLISION $\}$


TFNP
PLS
PPP
PPA
PPAD
P