Lecture 8
Nash Equilibrium Computation

CS 580

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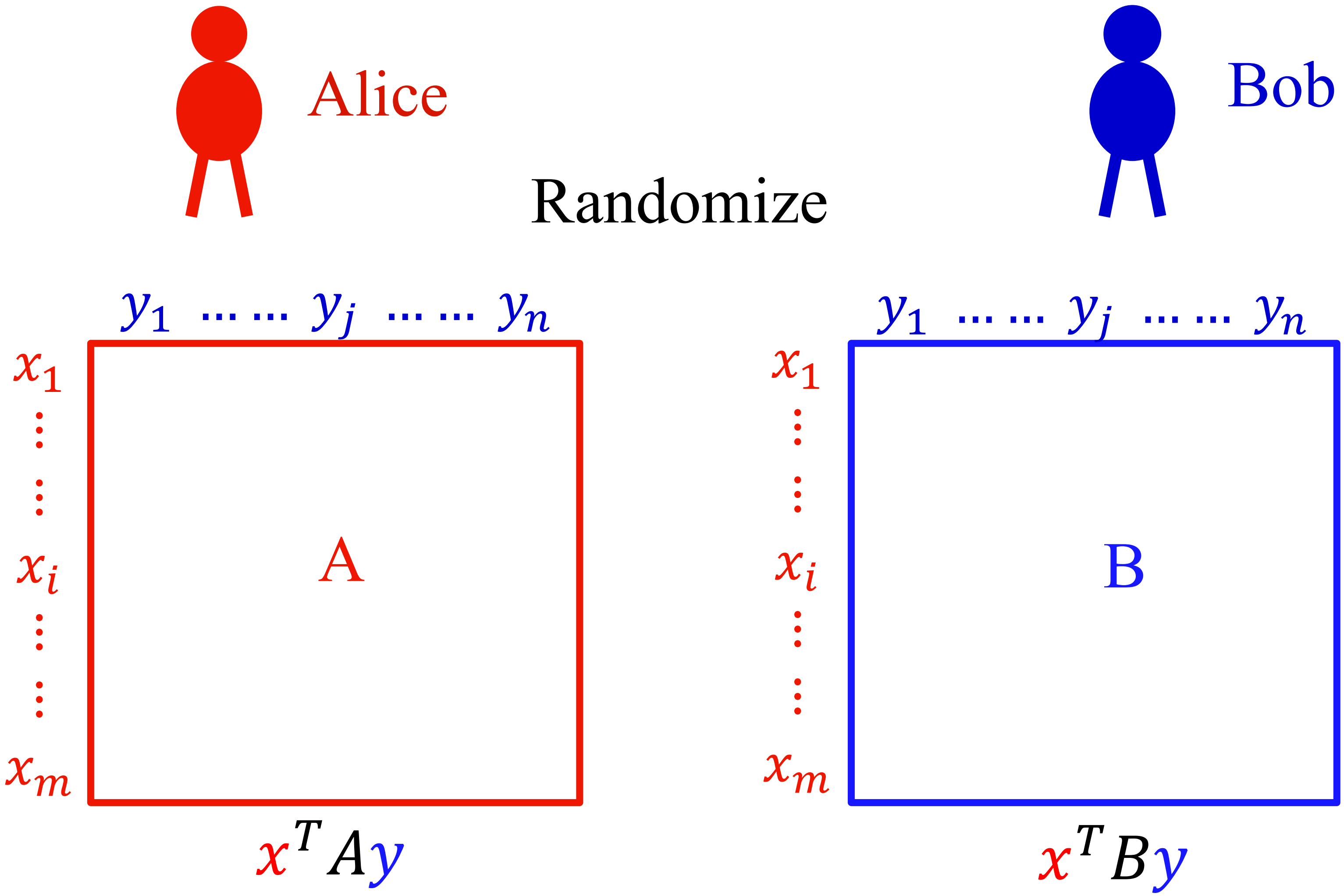
Recall

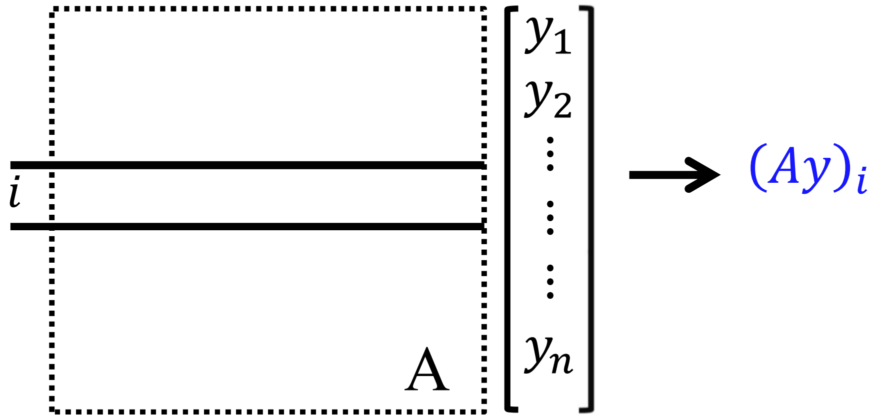


NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \qquad \forall z \in \Delta_m$$

$$x^T B y \geq x^T B z, \qquad \forall z \in \Delta_n$$

- i^{th} strategy gives Alice



- Max possible payoff: $\max_i e_i Ay$
- x achieves max payoff iff

$$\forall i, \qquad x^T Ay \geq (Ay)_i$$

$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow (Ay)_k = \max_i (Ay)_i$$

Complementarity

$$P \quad \boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$$

$$Q \quad \boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

2-Nash

max: $x^T(A+B)y - (\pi_A + \pi_B)$

s.t. $(y, \pi_A) \in P, (x, \pi_B) \in Q$

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$ Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

max: $x^T(0)y - (\pi_A + \pi_B)$

s.t. $(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

Theorem. [von Neumann'28] (max-min = min-max) Game $(A, -A)$
Wrt A, Alice is a maximizer and Bob minimizer

$$\max_x \min_y x^T A y = \min_y \max_x x^T A y$$ & the max-min is NE.

Proof assumed existence of NE

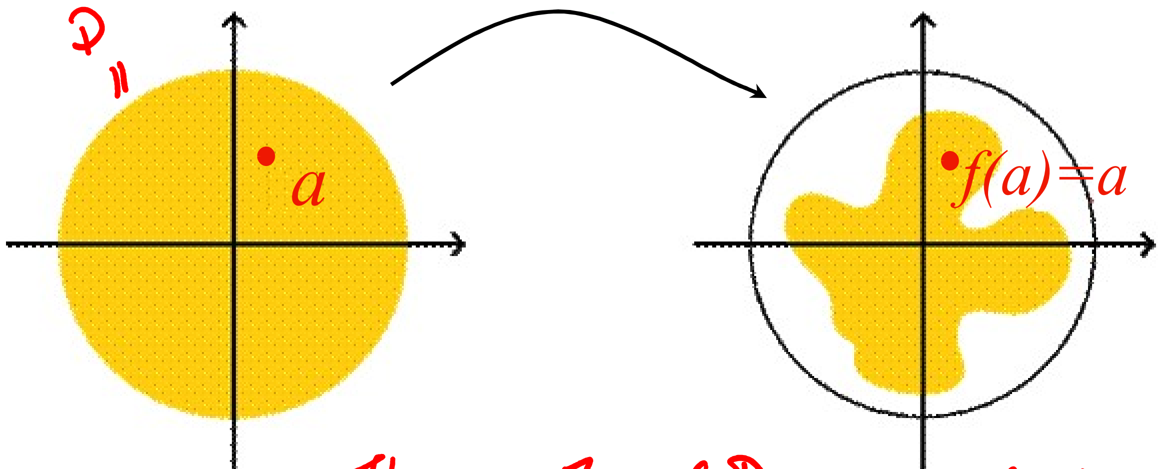
Nash's Proof of Existence

Uses Brouwer's fixed-point theorem

closed & bounded.
" compact , convex
D
"

continuous

$f : D \to D$



Then $\exists\, a \in D$ s.t. $f(a) = a$

a is a fixed-point of f.

Nash's Proof of Existence

$f: \Delta_m \times \Delta_n \to \Delta_m \times \Delta_\mathrm{n}$, $(x', y') = f(x, y)$

$\forall i,$ $\delta_i = \max\{(Ay)_i - x^T Ay, 0\},$

$\forall i,$ $x_i' = \dfrac{x_i + \delta_i}{\sum_k x_k + \delta_k}$

continuous func of (x, y)

Lemma. If $x' = x$ then x is best for Alice against y

$\equiv$ If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

$$\forall i, \qquad \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$$

$$\forall i, \qquad x_i' = \frac{x_i + \delta_i}{\sum_k x_k + \delta_k}$$

Lemma. If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

Pf:

$\min_k (Ay)_k \leq \sum_{i=1}^{m} x_i (Ay)_i = x^T Ay < \max_k (Ay)_k$

$\exists i : x_i > 0,$

$(Ay)_i < \max_k (Ay)_k$

$i^{b} = \arg\max_k (Ay)_k$

$i^{w} = \arg\min_{k: x_k > 0} (Ay)_k$

① $i^w \neq i^b$, ② $\delta_{i^w} = 0$, ③ $\delta_{i^b} > 0$

$x_{i^w} > 0$

Claim: $x'_{i^w} \neq x_{i^w}$

$x'_w = \frac{x_{i^w} + \delta_{i^w}}{\sum_k (x_k + \delta_k)} < x_{i^w}$ ($\delta_{i^w} = 0$ ($\because$ ②))

$\sum_k (x_k + \delta_k) \geq \sum_k x_k + \delta_{i^b} > 1$ ($\because$ ③)

Nash's Proof of Existence

$f: \Delta_m \times \Delta_n \rightarrow \Delta_m \times \Delta_\mathrm{n}$, $\quad (x', y') = f(x, y)$

$\forall j, \quad \tau_j = \max\left\{\left(x^T B\right)_j - x^T B y, 0\right\},$

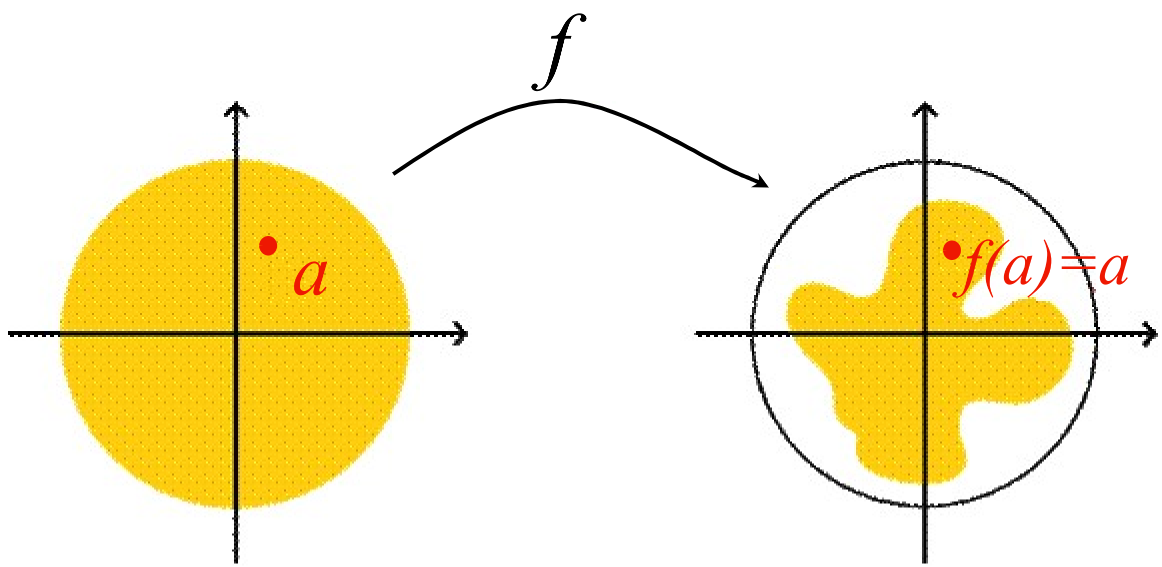
$$\forall j, \quad y_j' = \frac{y_j + \tau_j}{\sum_k y_k + \tau_k}$$

Lemma. If $y' = y$ then y is best for Bob against x

$\equiv$ If $x^T B y < x^T B z$ for some $z \in \Delta_n$ then $y' \neq y$.

Computation in general?

NE existence via fixed-point theorem.



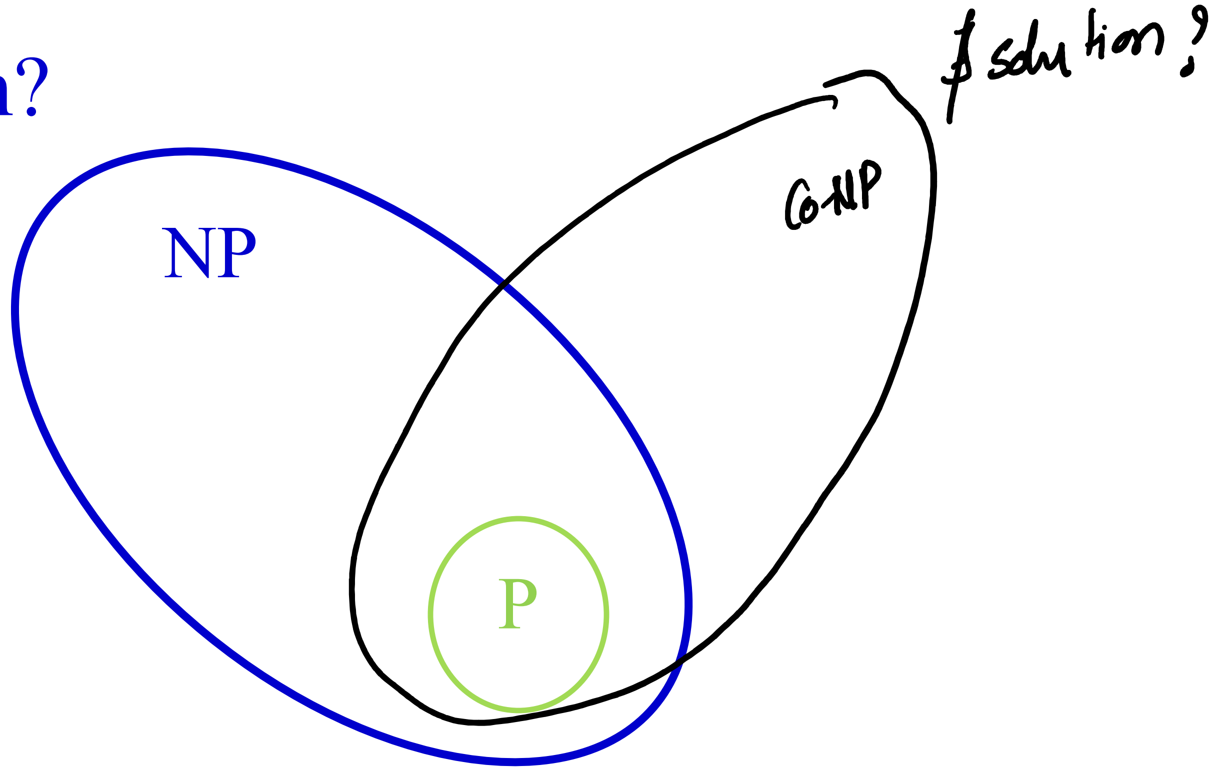
Computation? (in Econ)

- Special cases: Dantzig'51, Lemke-Howson'64, Elzen-Talman'88, Govindan-Wilson'03, …
- Scarf'67: Approximate fixed-point.
  - Numerical instability
  - Not efficient!
- …

Computation? (in CS)

Not easy!

∃ solution?



What if solution always exists, like Nash Eq.?

Computation? (in CS)

Megiddo and Papadimitriou'91 :

Nash is NP-hard $\Rightarrow$ NP=Co-NP

NP-hardness is ruled out!

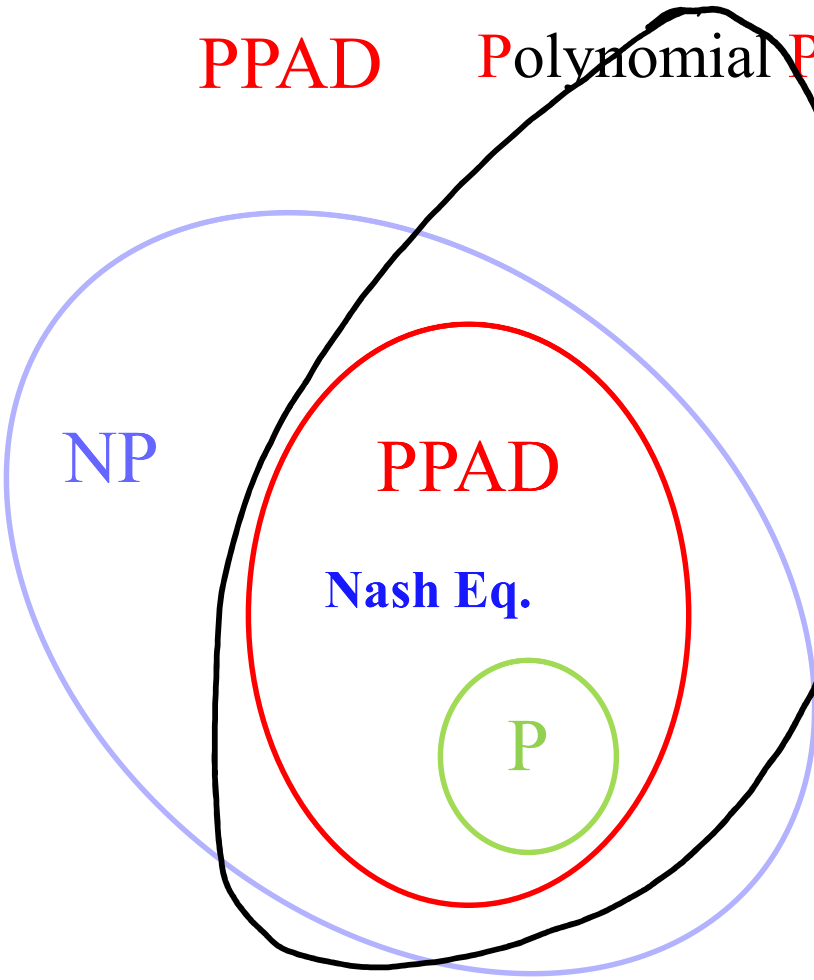
Complexity Classes

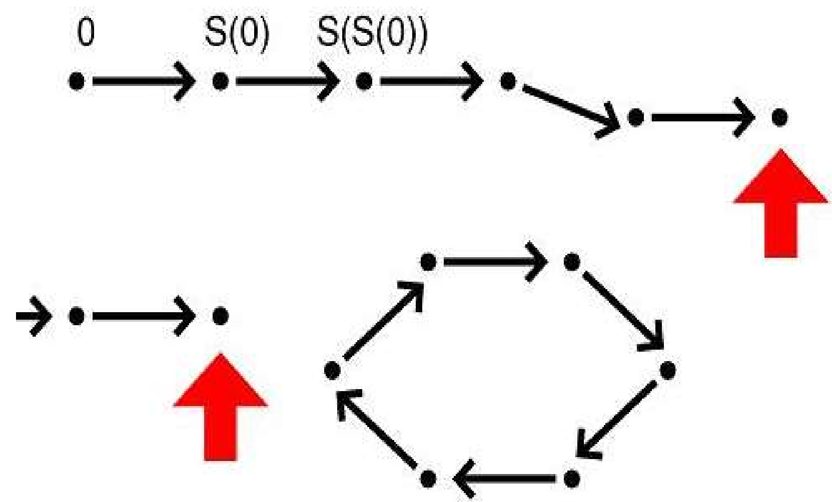
2-Nash is PPAD-complete!

[DGP'06, CDT'06]

Papadimitriou'94

PPAD Polynomial Parity Argument for Directed graph





Find an end

Brute-force Algorithm?

P: $\forall i, (Ay)_i \leq \pi_A$, $y \in \Delta_n$ ← max payoff of Alice against y.

Q: $\forall j, (x^T B)_j \leq \pi_B$, $x \in \Delta_m$

Let (x, y) be a NE. Suppose we know supp(x) and supp(y).

Now can we find a NE?

$S = \text{Supp}(x) = \{ i \mid x_i > 0 \} \neq \emptyset$

$T = \text{Supp}(y) = \{ j \mid y_j > 0 \} \neq \emptyset$

$LFP(S, T) \rightarrow$

$\forall i \notin S, \quad x_i = 0, \quad (Ay)_i \leq \pi_A$

$\forall j \notin T, \quad y_j = 0, \quad (x^T B)_j \leq \pi_B$

$\forall i \in S, \quad (Ay)_i = \pi_A, \quad x_i \geq 0$

$\forall j \in T, \quad (x^T B)_j = \pi_B, \quad y_j \geq 0$

$\sum_{i=1}^{m} x_i = 1, \quad \sum_{j=1}^{n} y_j = 1$

Can we do better?

Not so far. And may be never!

It is one of the hardest problems in PPAD.

What about special cases/approximation?

- Rank(A) or rank(B) is constant
- O(1)-approximate NE: quasi-polynomial time algorithm
- Constant rank games: rank(A+B) is a constant
  - FPTAS

Nash equilibrium: Scale Invariance

Nash equilibrium (NE)

$$x^T A y \geq z^T A y, \quad \forall z$$

$$x^T B y \geq x^T B z, \quad \forall z$$

$$A' = \alpha A + c, \qquad B' = \beta B + d, \qquad \alpha, \beta \geq 0$$

Claim. (x, y) NE for the game (A', B') as well.

Approximate Nash equilibrium

Wlog. $(A, B) \in [0,1]^{2mn}$

ϵ-NE:

$$x^T A y \geq z^T A y - \epsilon, \qquad \forall z$$
$$x^T B y \geq x^T B z - \epsilon, \qquad \forall z$$

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$

Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

2-Nash

max: $x^T(A+B)y - (\pi_A + \pi_B)$

s.t. $(y, \pi_A) \in P, (x, \pi_B) \in Q$

Problematic

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n\end{array}}$

Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \leq \pi_B \\ x \in \Delta_m\end{array}}$

$$C = A + B$$

2-Nash

max: $x^T C y - (\pi_A + \pi_B)$

s.t. $(y, \pi_A) \in P, (x, \pi_B) \in Q$

$\longrightarrow$

LP(δ)

max: $x^T \delta - (\pi_A + \pi_B)$

s.t. $(y, \pi_A) \in P, (x, \pi_B) \in Q$

$\delta_i \leq (Cy)_i, \quad \forall i \leq m$

LP(δ)

$$\max: \; x^T\delta - (\pi_A + \pi_B)$$
$$\text{s.t.} \; \forall i, (Ay)_i \le \Pi_A; \quad y \in \Delta_n$$
$$\forall j, \left(x^T B\right) \le \Pi_B; \quad x \in \Delta_m$$
$$\delta_i \le (Cy)_i, \qquad \forall i \le m$$

OPTVal(δ) = opt value of LP(δ)

OPT(δ) = sol^{n}s of LP(δ)

Lemma 1. $\forall \delta \in R^m$, OPTVal(δ)≤ 0

$$x^T\delta \le x^T(Cy) = x^T Ay + x^T By \le \pi_A + \pi_B$$

$$\Rightarrow x^T\delta - \pi_A - \pi_B \le 0 \qquad \forall (x, y, \pi_A, \pi_B) \text{ feasible in LP}(\delta)$$

LP(δ)

$$\max: \; x^T\delta - (\pi_A + \pi_B)$$

$$\text{s.t. } \forall i, (Ay)_i \leq \Pi_A; \quad y \in \Delta_n$$

$$\forall j, \left(x^T B\right) \leq \Pi_B; \quad x \in \Delta_m$$

$$\delta_i \leq (Cy)_i, \qquad \forall i \leq m$$

OPTVal(δ)

OPT(δ)

Lemma 2. If OPTVal(δ)$\geq -\epsilon$, then OPT(δ) forms ϵ-NE

Pf: Let $(x, y, \pi_A, \pi_B) \in OPT(\delta)$

Then $x^T A y \leq \pi_A$, $x^T B y \leq \pi_B \rightarrow$ (1)

$$-\epsilon \leq OPTVal(\delta) = x^T\delta - (\pi_A + \pi_B)$$
$$\leq x^T(Cy) - \pi_A - \pi_B$$
$$= (x^T A y - \pi_A) + (x^T B y - \pi_B)$$

(1) $\Rightarrow$ $x^T A y \geq \pi_A - \epsilon$, $x^T B y \geq \pi_B - \epsilon$

$\Rightarrow x^T A y \geq z^T A y - \epsilon \; \forall z \in \Delta_m$; $x^T B y \geq x^T B z - \epsilon \quad \forall z \in \Delta_n$

$\Rightarrow (x, y)$ is ϵ-NE ∎

$$\max: \ x^T\delta - (\pi_A + \pi_B)$$

LP(δ)

$$\text{s.t.} \ \forall i, (Ay)_i \le \Pi_A; \quad y \in \Delta_n$$
$$\forall j, (x^T B) \le \Pi_B; \quad x \in \Delta_m$$
$$\delta_i \le (Cy)_i, \qquad \forall i \le m$$

OPTVal(δ)

OPT(δ)

Lemma 3. $\exists \delta$ such that OPTVal(δ) $\ge -\epsilon$

Pf: Let (x^*, y^*) be a NE. $\pi_A^* = x^{*T} A y^*$, $\pi_B^* = x^* B y^*$

Set δ^* s.t. $\forall i$, $(Cy^*)_i - \epsilon \le \delta_i^* \le (Cy^*)_i$

Then $(x^*, y^*, \pi_A^*, \pi_B^*)$ is feasible in $LP(\delta^*)$

AND, $\text{OPTVal}(\delta^*) \ge x^{*T}\delta^* - \pi_A^* - \pi_B^*$

$\ge x^{*T}(Cy^* - \epsilon) - \pi_A^* - \pi_B^*$

$= \underbrace{x^* C y^* - \pi_A^* - \pi_B^*}_{=\,0 \ (\because \text{NE})} - \epsilon$

$= -\epsilon$

∎

ϵ-NE: Lipton-Markakis-Mehta'03

We know: (x^*, y^*) NE. Then it suffices to have

$$(Cy^*)_i - \epsilon \le \delta_i \le (Cy)_i + \epsilon \quad \forall i$$

Good-Event

Idea: Show that there is a sparse such δ. And enumerate.

High level Approach: Find y s.t. $|(Cy^*)_i - (Cy)_i| < \epsilon$ & set $\delta = Cy$

→ S: a multi set, $S \subseteq \{1, \ldots, n\}$, $|S| = k$

$y = \frac{S}{k}$ a.k.a. $\forall j: y_j := \frac{\#\text{ times } j \text{ in } S}{k}$

→ Enumerate over all S & check.

$O(n^k)$ $\delta = C\left(\frac{S}{k}\right)$ if OPTVal$(\delta) \ge -\epsilon$

Q: Why such S exists? & what size (k)?

Algo: ① $d = 1 \ldots k$, Sample $j_d \in \{1 \ldots n\} \sim y^*$: $j_d = j$ w.p. y^*_j $\left(k = \frac{\log m}{\epsilon^2} + 1\right)$

(Given y^*)

② $S = \{j_1, \ldots, j_k\}$ (multi-set)

③ $\delta_j = \frac{\#\text{ times } j \in S}{k}$; $\delta = cy$

④ Output OPT(δ) (ϵ-NE)

Correctness Proof: Suffices to show $|(cy)^*_i - (cy)_i| \leq \epsilon, \forall i$

Fix $i \in \{1 \ldots m\}$.

Define (identical) Random Variable $X_1 \ldots X_k$ where $X_d = c_{i j_d}$. That is $X_d = c_{ij}$ w.p. y^*_j

$$\bar{X} = \frac{\sum_{d=1}^{k} X_i}{k} \Rightarrow \bar{X} = (cy)_i$$

$$\forall l,\ \mathbb{E}[X_l] = \sum_{j=1}^{n} c_{ij} y^*_j = (Cy^*)_l \Rightarrow \mathbb{E}[\bar{X}] = (Cy^*)_i$$

By Hoeffding's inequality

$$\Pr\left[\left|\bar{X} - \mathbb{E}[\bar{X}]\right| > \epsilon\right] < 2e^{-\frac{k^2\epsilon^2}{k}} = 2e^{-k\epsilon^2}$$

$$\Downarrow$$

$$\Pr\left[\left|(Cy)_i - (Cy^*)_i\right| > \epsilon\right] < 2e^{-k\epsilon^2},\ \forall i.$$

$$\Rightarrow \Pr[\text{Bad-event}] = \Pr\left[\exists i : \left|(Cy)_i - (Cy^*)_i\right| > \epsilon\right]$$

$$\leq \sum_{i=1}^{m} \Pr\left[\left|(Cy)_i - (Cy^*)_i\right| > \epsilon\right] \leq 2me^{-k\epsilon^2}$$

$$\left(\because k > \frac{\lg m}{\epsilon^2}\right) < 1$$

$\Rightarrow$ $\Pr[\text{Good-event}] = \Pr\left[\forall i : |(Cy)_i - (Cy^*)_i| \leq \epsilon\right]$

$= 1 - \Pr[\text{Bad-event}] > 0$

$\Rightarrow$ $\exists$ S s.t. $|S| = k$, S is a multiset of $\{1 \ldots n\}$,

set for $y = \frac{S}{K}$ we have

$\forall i : |(Cy)_i - (Cy^*)_i| \leq \epsilon$

$\Rightarrow$ Enumerating over S suffices. ∎

Constant Rank Games

P $\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}$

Q $\begin{array}{c} \forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m \end{array}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\max: -(\pi_A + \pi_B)$$
$$\text{s.t. } (y, \pi_A, x, \pi_B) \in P \times Q$$

Rank of a game: rank(A+B)

Zero-sum ≡ Rank-0 games

Rank-0 (zero-sum) games rank(A+B)=0 → Von Neumann (1928) → LP

Rank-1 games rank(A+B)=1 → λ → $\underline{x} = \mathrm{LP}(\lambda)$ → $\underline{c}^T \underline{x}$

Rank-2 games rank(A+B)=2 → λ_1, λ_2 → $\underline{x} = \mathrm{LP}(\lambda_1, \lambda_2)$ → $\underline{c}^T \underline{x}$, $\underline{d}^T \underline{x}$

⋮

⋮

Rank-0 (zero-sum) games → (Von Neumann (1928)) LP

Rank-1 games → 1-D Fixed Point

In P

Rank-2 games → 2-D Fixed Point

⋮

Rank-0 (zero-sum) games → LP
Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general → 2-D Fixed Point

Rank-2 games → 2-D Fixed Point

⋮

⋮

Rank-0 (zero-sum) games → LP
Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general

Rank-2 games ← ? 2-D Fixed Point

⋮

⋮

Rank-0 (zero-sum) games → LP

Von Neumann (1928)

Rank-1 games → 1-D Fixed Point

PPAD-hard in general

Rank-2 games ← 2-D Fixed Point

[M.'14, COPY'16]

⋮

⋮