Lecture 7
Games and Nash Equilibrium

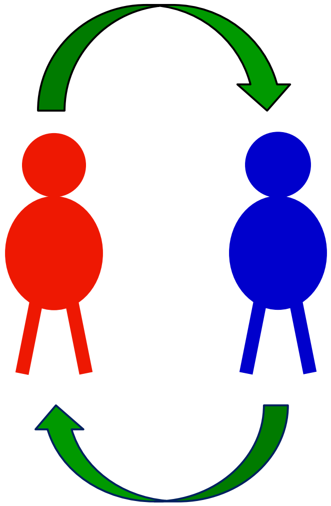
CS 580

14^{th} September 2021

Instructor: Ruta Mehta

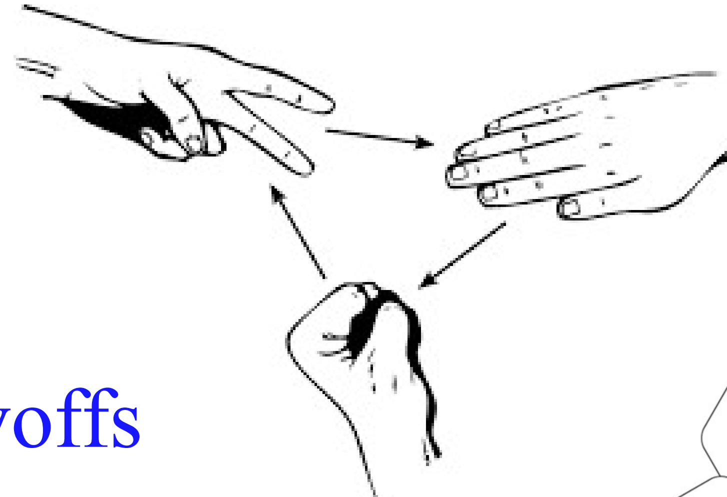
ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Games

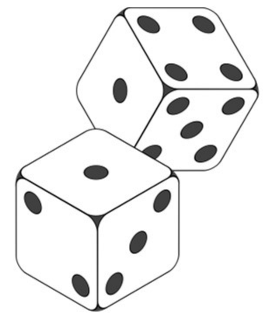


Players

Payoffs

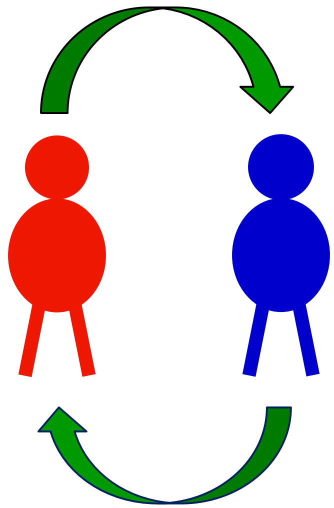


Strategies

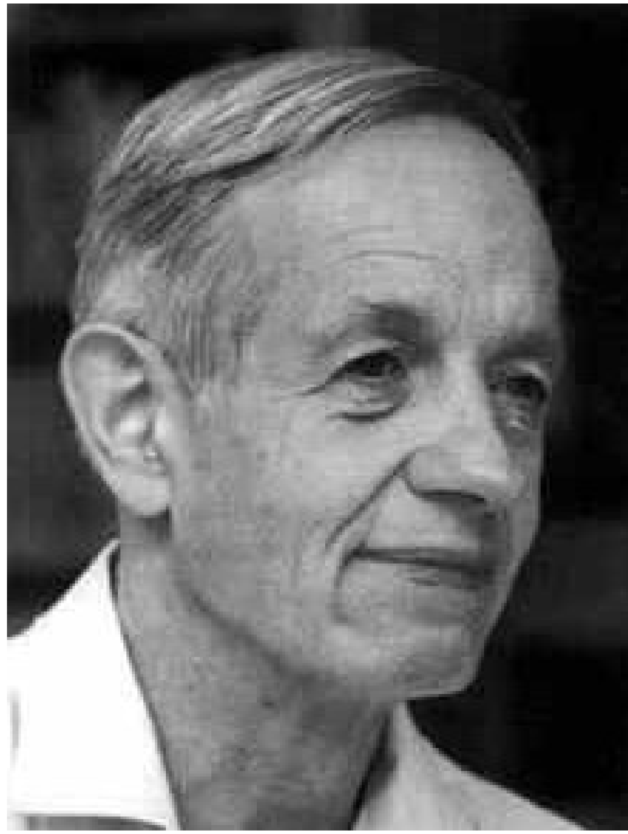


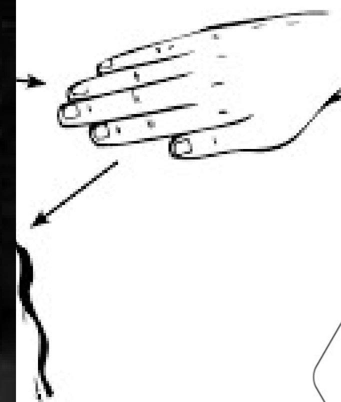
Randomize!

Games

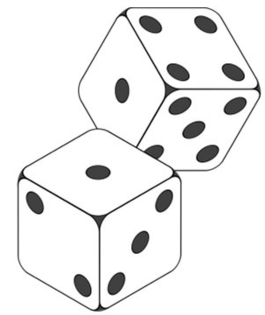


Players





tegies



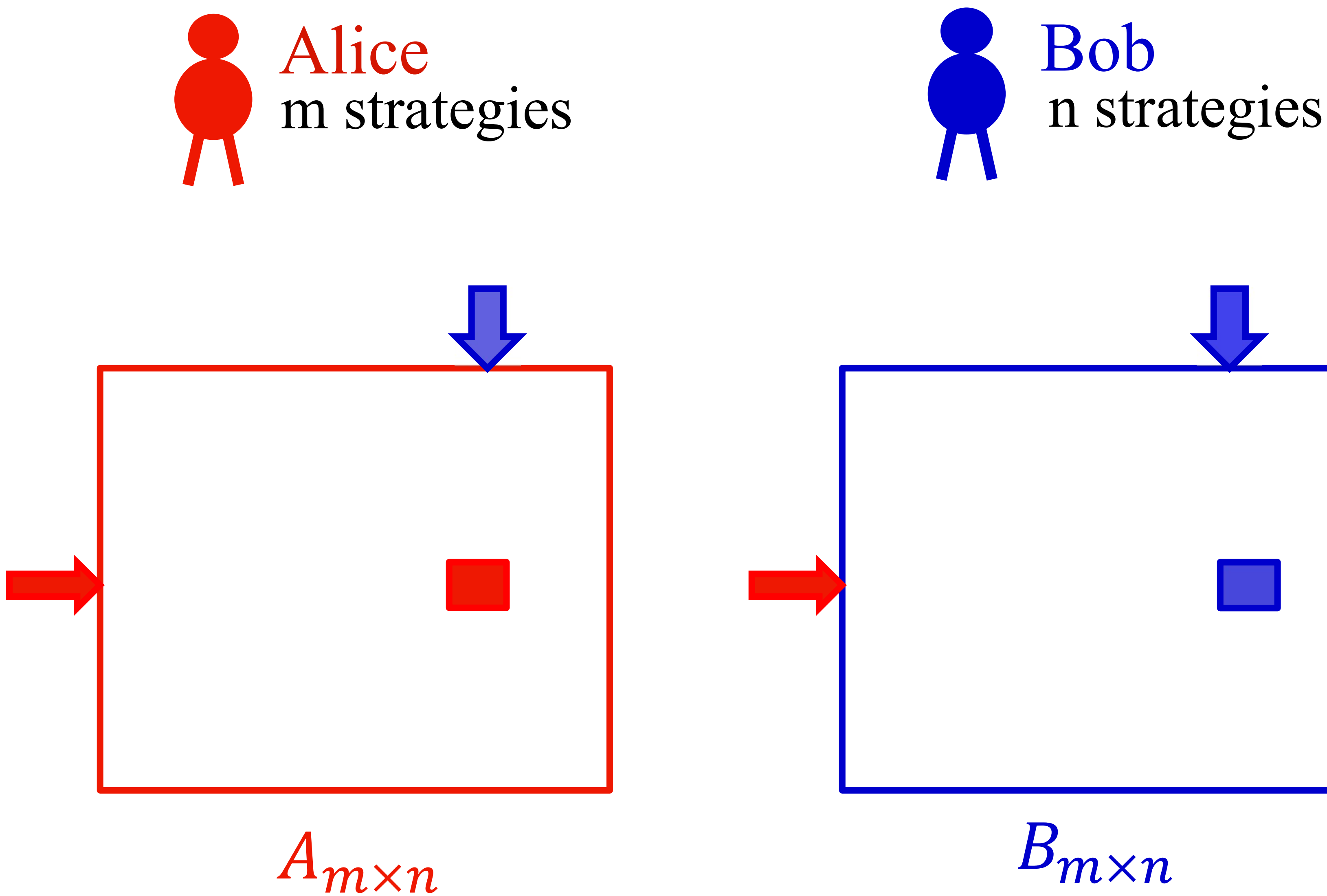
Randomize!

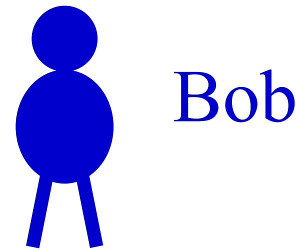
Nash (1950):

There exists a (stable) state where no player gains by unilateral deviation.

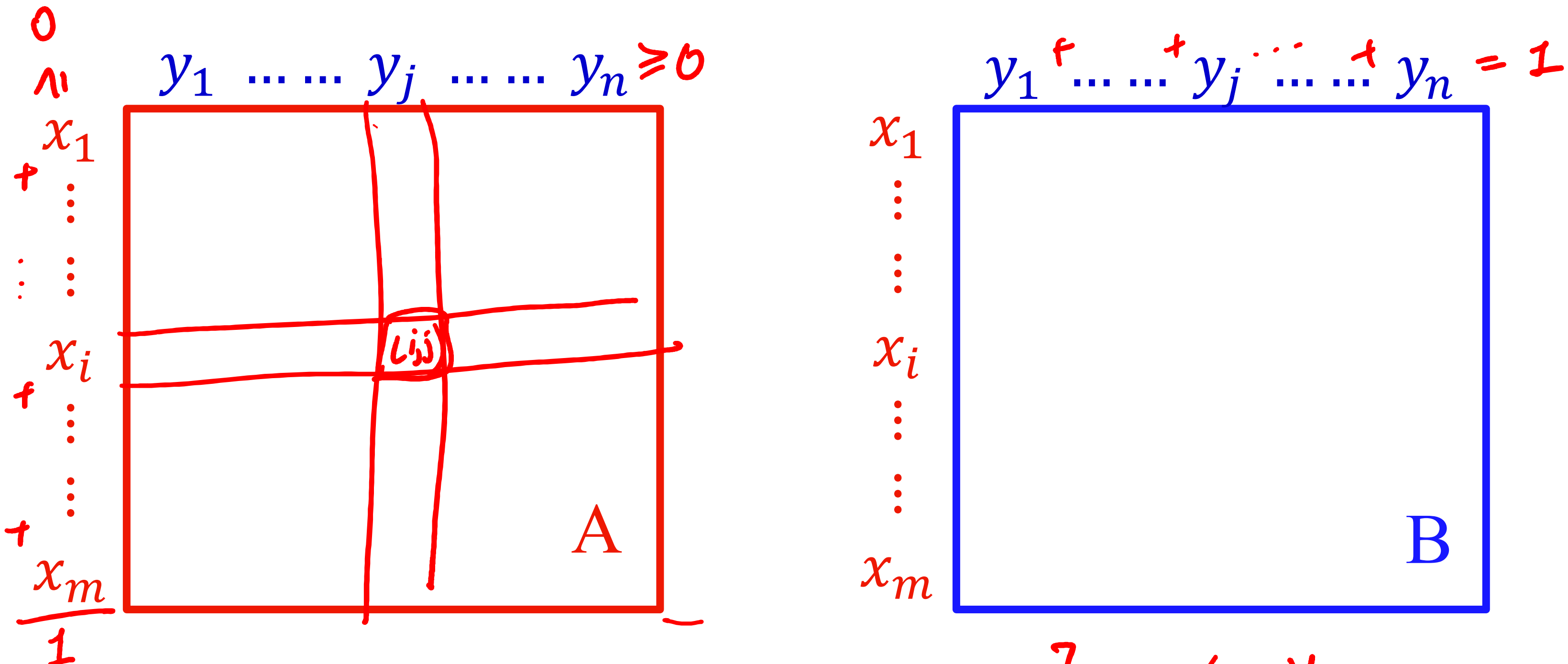
Nash equilibrium (NE)

Our focus: Two-player games





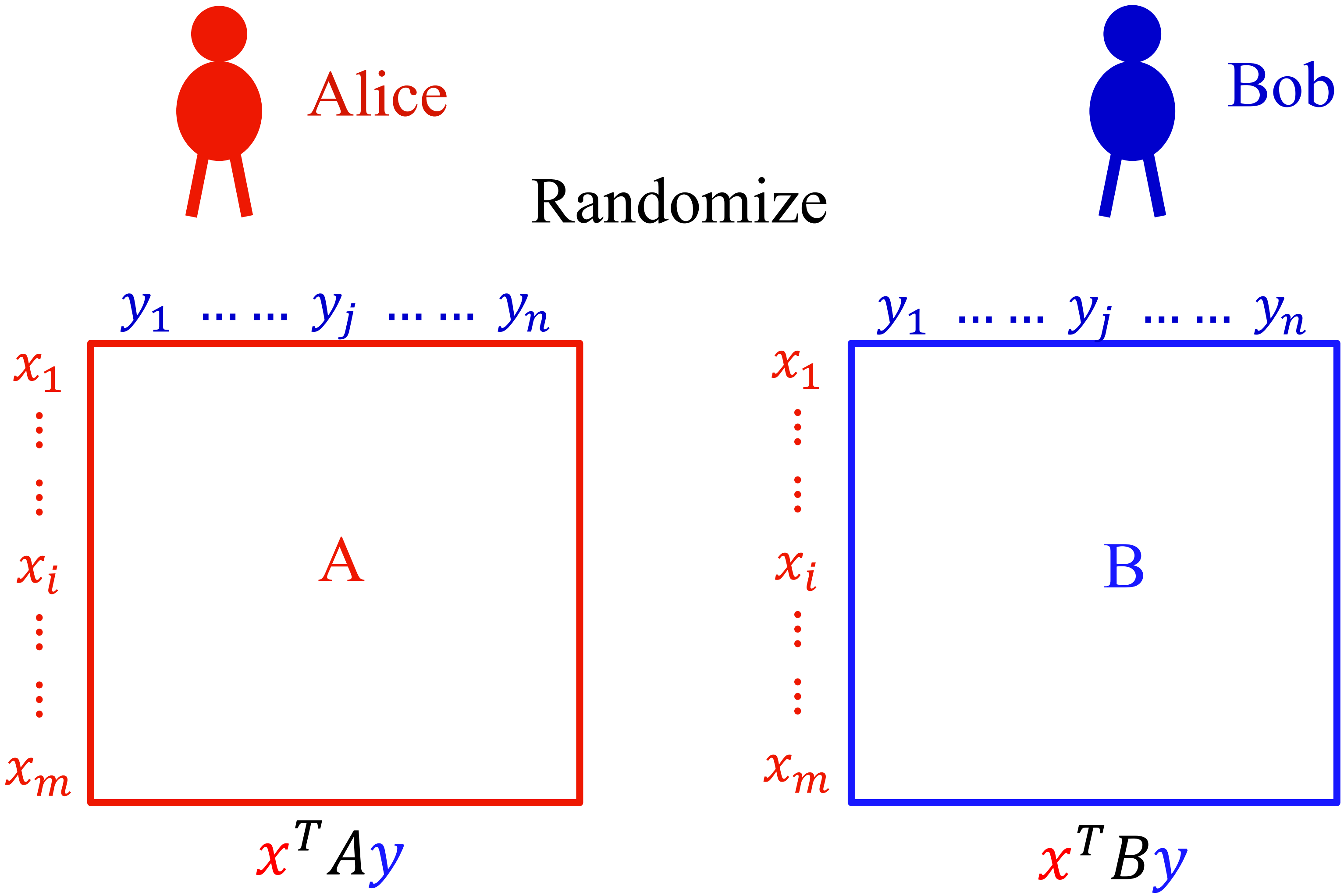
Randomize



$\Pr[\text{Alice plays } i \text{ AND Bob plays } j] = x_i y_j$

with this prob Alice gets A_{ij} payoff

$$\mathbb{E}[\text{Payoff of Alice}] = \sum_{i,j} (x_i \cdot y_j) \cdot A_{ij} = \sum_i x_i \sum_j A_{ij} y_j = x^T A y$$



NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \qquad \forall z \in \Delta_m = \{z \in \mathbb{R}^m \mid z \geq 0, \sum_{i=1}^{m} z_i = 1\}$$

$$x^T B y \geq x^T B z, \qquad \forall z \in \Delta_n$$

Example

	R ($\frac{1}{3}$, $\frac{1}{4}$)	P ($\frac{1}{3}$, $\frac{1}{2}$)	S ($\frac{1}{3}$, $\frac{1}{4}$)
R ($\frac{1}{3}$)	0 0 ($\frac{1}{12}$)	-1 1 ($\frac{1}{6}$)	1 -1 ($\frac{1}{12}$)
P ($\frac{1}{3}$)	1 -1 ($\frac{1}{12}$)	0 0 ($\frac{1}{6}$)	-1 1 ($\frac{1}{12}$)
S ($\frac{1}{3}$)	-1 1 ($\frac{1}{12}$)	1 -1 ($\frac{1}{6}$)	0 0 ($\frac{1}{12}$)

E[payoff of Alice]

$= \frac{1}{12}(0+1-1) + \frac{1}{6}(-1+0+1) + \frac{1}{12}(1-1+0)$

$= 0$

E[payoff of Bob] $= 0$

For Alice:

$E[\text{payoff from } R] = 0\cdot\frac{1}{4} + (-1)\frac{1}{2} + 1\left(\frac{1}{4}\right) = -\frac{1}{4}$

$E[\;''\quad''\; P] = \frac{1}{4} + 0 - \frac{1}{4} = 0$

$E[\;''\quad''\; S] = -\frac{1}{4} + \frac{1}{2} + 0 = \frac{1}{4}$

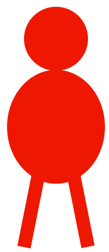
$\max E[\;''\quad''\; (x_1, x_2, x_3)] = \left(-\frac{1}{4}\right)x_1 + (0)\,x_2 + \left(\frac{1}{4}\right)\cdot x_3$

$x_1 \to 0$, $x_2 \to 0$, $x_3 \to 1$

(randomize) play only the moves that give max-payoff

2-Nash Characterization

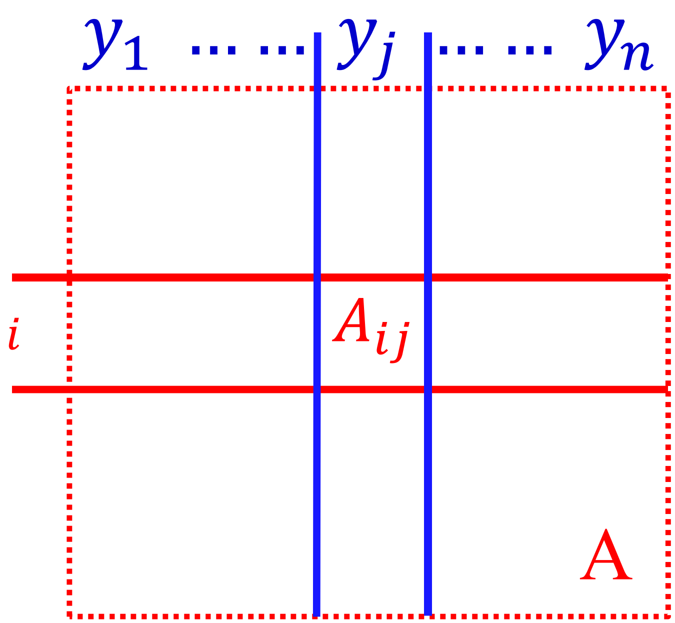
2-Nash Characterization



Alice

Bob

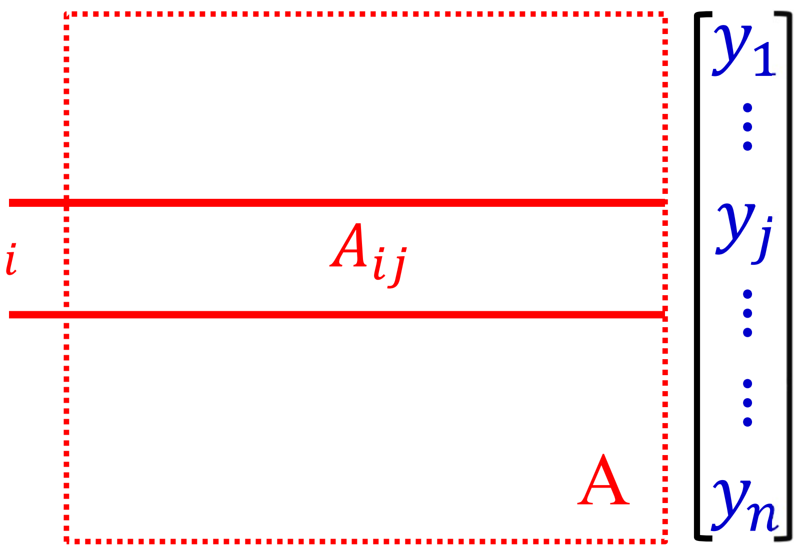
- For Row, i^{th} strategy gives



$$\longrightarrow \sum_j A_{ij} y_j$$

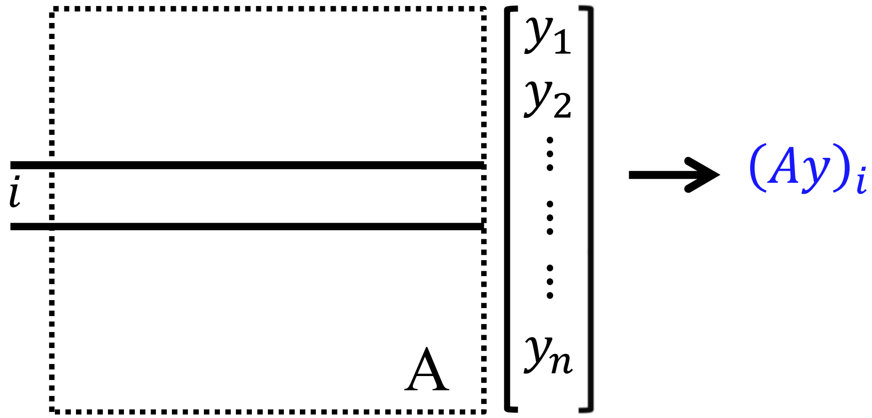
2-Nash Characterization

- For Row, i^{th} strategy gives



$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

- i^{th} strategy gives Alice



- Max possible payoff: $\max_i e_i Ay$
- x achieves max payoff iff

$$\forall i, \qquad x^T Ay \geq (Ay)_i$$

$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow (Ay)_k = \max_i (Ay)_i$$

Complementarity

- Max possible payoff: $\max_i e_i A y$
- x achieves max payoff iff

$$\forall i, \qquad x^T A y \geq (Ay)_i$$

$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow (Ay)_k = \max_i (Ay)_i$$

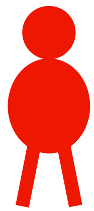
Complementarity

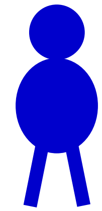
A	R ($\frac{1}{2}$)	P (0)	S ($\frac{1}{2}$)
x_1 R	0	-1	1
x_2 P	1	0	-1
x_3 S	-1	1	0

$$y = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} \qquad Ay = \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \begin{matrix} \leftarrow R \\ \leftarrow P \\ \leftarrow S \end{matrix}$$

$$\underset{x}{\arg\max} \left(x_1 \cdot \tfrac{1}{2} + x_2 \cdot 0 + x_3 \left(-\tfrac{1}{2}\right) \right) = (1,0,0) \qquad \max_x \left(x_1 \cdot \tfrac{1}{2} + x_2 \cdot 0 + x_3 \left(-\tfrac{1}{2}\right) \right) = \tfrac{1}{2}$$

Polyhedra





max-payoff $\leq \pi_A$

max-payoff $\leq \pi_B$

P: $\forall i, (Ay)_i \leq \pi_A$, $y \in \Delta_n$

Q: $\forall j, \left(x^T B\right)_j \leq \pi_B$, $x \in \Delta_m$

$(y, \pi_A) \in P$, Bob is fixed to y

$\Rightarrow \max_i (Ay)_i \leq \pi_A$

Suppose Alice plays $x \in \Delta_m$

$x^T A y \leq \max_i (Ay)_i \leq \pi_A$

$(x, \pi_B) \in Q$

$\max_j (x^T B)_j \leq \pi_B$

If Bob plays y

$x^T B y \leq \pi_B$

$x^T A y + x^T B y \leq \pi_A + \pi_B$

P

$$\forall i, (Ay)_i \leq \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, \left(x^T B\right)_j \leq \pi_B$$
$$x \in \Delta_m$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Sum of payoffs

At least the sum of max payoffs

$$\underbrace{x^T (A+B) y}_{\text{Sum of payoffs}} - \underbrace{(\pi_A + \pi_B)}_{\text{At least the sum of max payoffs}} \leq 0$$

$$x^T A y + x^T B y$$

P $\forall i, (Ay)_i \le \pi_A$, $y \in \Delta_n$

Q $\forall j, \left(x^T B\right)_j \le \pi_B$, $x \in \Delta_m$

$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

Sum of payoffs: $x^T(A+B)y$

At least the sum of max payoffs: $(\pi_A + \pi_B)$

$$\underbrace{x^T(A+B)y}_{\text{Sum of payoffs}} - \underbrace{(\pi_A + \pi_B)}_{\text{At least the sum of max payoffs}} = \mathbf{0}$$

$\Updownarrow$

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$ $\Rightarrow (y, \pi_A)$: $x^T A y \le \pi_A$ $\forall x \in \Delta_m$ $\Rightarrow$ ①

Q $\boxed{\begin{array}{c}\forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m\end{array}}$ $\Rightarrow (x, \pi_B)$: $x^T B y \le \pi_B$ $\forall y \in \Delta_n$ $\Rightarrow$ ②

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T(A+B)y - (\pi_A + \pi_B) \le 0.$ ✓ ($\because$ ①, ②)

(ii) $x^T(A+B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

$$\underbrace{(x^T A y - \pi_A)}_{\le 0} + \underbrace{(x^T B y - \pi_B)}_{\le 0} = 0$$

$\Leftrightarrow$ $x^T A y = \pi_A$, $x^T B y = \pi_B$ ($\because$ $x^T A y \le \pi_A$, $x^T B y \le \pi_B$)

$\Leftrightarrow$ $x^T A y = \pi_A \ge \max_{z \in \Delta_m} z^T A y$ ($\because$ ①)

$x^T B y = \pi_B \ge \max_{z \in \Delta_n} x^T B z$ ($\because$ ②)

$\Leftrightarrow$ (x, y) is NE.

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n\end{array}}$

Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \leq \pi_B \\ x \in \Delta_m\end{array}}$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T(A + B)y - (\pi_A + \pi_B) \leq 0$.

(ii) $x^T(A + B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$

Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

$(y, \pi_A) \in P, \quad (x, \pi_B) \in Q$

2-Nash

Sum of payoffs

At least the sum of max payoffs

max: $\overbrace{x^T(A+B)y}^{\text{Sum of payoffs}} - \overbrace{(\pi_A + \pi_B)}^{\text{At least the sum of max payoffs}}$ $\boxed{= \mathbf{0}}$

s.t. $(y, \pi_A) \in P, (x, \pi_B) \in Q$

⇕

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

P $\forall i, (Ay)_i \leq \pi_A$, $y \in \Delta_n$

Q $\forall j, \left(x^T B\right)_j \leq \pi_B$, $x \in \Delta_m$

$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\text{max: } x^T (A + B) y - (\pi_A + \pi_B)$$

$$\text{s.t. } (y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

$$P \quad \boxed{\begin{array}{c} \forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n \end{array}} \qquad\qquad Q \quad \boxed{\begin{array}{c} \forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m \end{array}}$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\text{max: } -(\pi_A + \pi_B)$$
$$\text{s.t. } (y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. [von Neumann'28] (max-min = min-max) Game $(A, -A)$

Wrt A, Alice is a maximizer and Bob minimizer

x y $(x^TAy, -x^TAy)$

$$\max_x \min_y x^TAy = \min_y \max_x x^TAy$$ & the max-min is NE.

Pf: $x^* = \arg\max_x \min_y x^TAy$, $y^* = \arg\min_y \max_x x^TAy$

$$\max_x \min_y x^TAy = \min_y x^{*T}Ay \le x^{*T}Ay^* \le \max_x x^TAy^* = \min_y \max_x x^TAy$$

$=$ $=$ $\Rightarrow (x^*, y^*)$ is a NE.

$\Uparrow$

Claim: $(\tilde{x}, \tilde{y})$ is NE

$$\min_y x^{*T}Ay \ge \tilde{x}^TA\tilde{y} \ge \max_x x^TAy^*$$

$(\tilde{x}, \tilde{y})$ is a NE $\Rightarrow$ $\tilde{x}^T A \tilde{y} = \max_{x \in \Delta_m} x^T A \tilde{y} \to (1)$

$\tilde{x}^T A \tilde{y} = \min_{y \in \Delta_n} \tilde{x}^T A y \to (2)$

$\min_y x^{*T} A y \geq \min_y \tilde{x}^T A y$ ($\because$ By def. of x^*)

$= \tilde{x}^T A \tilde{y}$ ($\because$ (2))

$\max_x x^T A y^* \leq \max_x x^T A \tilde{y}$ ($\because$ By def y^*)

$= \tilde{x}^T A \tilde{y}$ ($\because$ (1))

$\Rightarrow \min_y x^{*T} A y \geq \tilde{x}^T A \tilde{y} \geq \max_x x^T A y^*$