Atomic Routing

Wednesday, November 3, 2021 11:06 AM

→ Directed n/w $G=(V,E)$, $e \in E$, $c_e : \mathbb{N} \to \mathbb{R}_+$
non-neg, non-decreasing.

→ Agents $N = \{1, \dots, n\}$

→ $i \in N$, wants to go from s_i to t_i, $s_i, t_i \in V$

$\mathcal{P}_i$ = all $s_i - t_i$ paths.

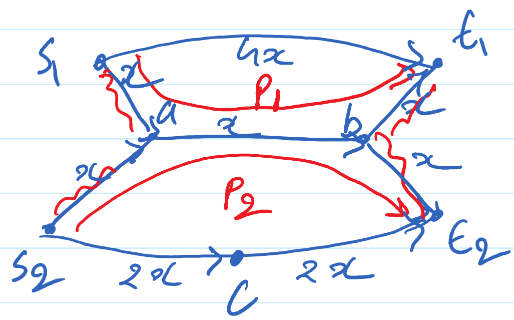
$P_i \in \mathcal{P}_i$, $P = (P_1, \dots, P_n) \in \bigtimes_i \mathcal{P}_i = \mathcal{P}$.

→ f_e^P = # agents taking edge e in P. ⇒ cost on e is $c_e(f_e^P)$
(f_e)

→ $c_i(P) = \sum_{e \in P_i} c_e(f_e^P)$ $\text{cost}(P) = \sum_i c_i(P)$

Example:

$P = (P_1, P_2)$



$\mathcal{P}_1 = \{s_1 - t_1,\ s_1 - a - b - t_1\}$

$\mathcal{P}_2 = \{s_2 - c - t_2,\ s_2 - a - b - t_2\}$

$f^P_{(s_1,t_1)} = f_{(s_2,a)} = f_{(s_1,a)} = f_{(b,t_2)} = 1$, $f_{(s_2,c)} = f_{(c,t_2)} = 0$
$= f_{(b,t_1)}$

$f_{(a,b)} = 2$

$c_1(P_1, P_2) = c_{(s_1,a)}(1) + c_{(a,b)}(2) + c_{(b,t_1)}(1)$

$= \quad 1+2+1=4$

$$\boxed{\begin{array}{l}\text{NE}: \; P=(P_1,\dots,P_n) \text{ is at a NE iff} \\ \forall i: \; C_i(P) \le C_i(Q_i, P_{-i}) \;, \; \forall Q_i \in \mathcal{P}_i\end{array}}$$

Potential Games.

✡ Potential Func.

$$\phi(P) = \sum_{e \in E} \underline{\sum_{k=1}^{f_e^P} c_e(k)}$$

Prop: ϕ is a potential Func.

$\forall i, \; \forall P \in \mathcal{P}, \quad C_i(Q_i, P_{-i}) - C_i(P_i, P_{-i}) = \phi(Q_i, P_{-i}) - \phi(P_i, P_{-i})$

$\forall Q_i \in \mathcal{P}_i$.

✡ Consequences (Potential Game)

① Pure NE

$\operatorname{argmin}_{P \in \mathcal{P}} \phi(P)$ is a NE.

② Simple local algo to find a NE.

(1) start with some $P \in \mathcal{P}$

② While P is not a NE

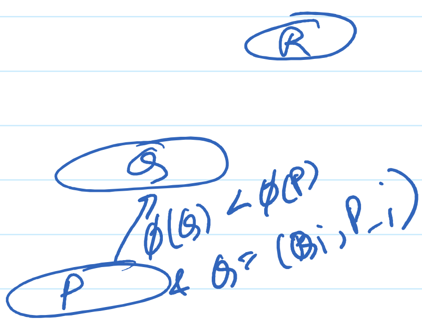
$\exists i, Q_i$ s.t. $\phi(Q_i, P_{-i}) < \phi(P)$

$P = (Q_i, P_{-i})$

③ Finding Pure NE is in PLS : Given DAG on exponentially many vertices through line cit rep. Find a sink.

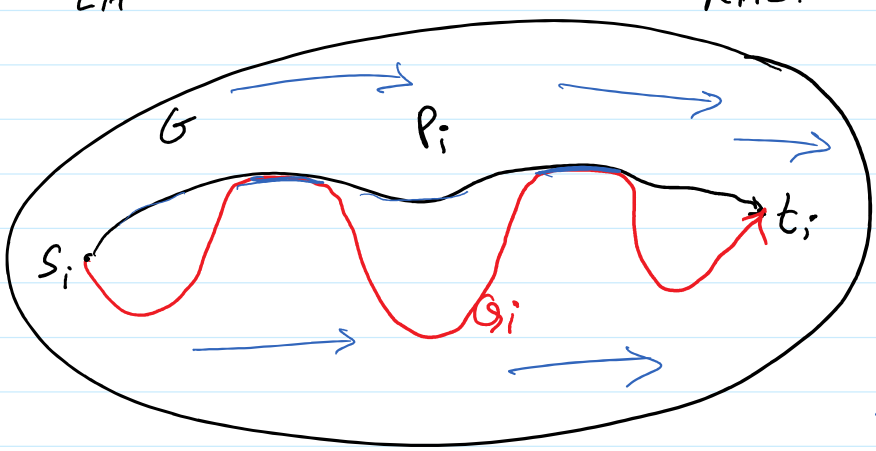
$V^{DAG} = \mathcal{P}$

$P, Q, R \in \mathcal{P}$.



Proof of the Prop : Fix, i, $P(\mathcal{P})$ $Q_i \in \mathcal{P}_i$

$$\underbrace{C_i(Q_i, P_{-i}) - \boxed{C_i(P_i, P_{-i})}}_{LHS} = \underbrace{\phi(Q_i, P_{-i}) - \phi(P_i, P_{-i})}_{RHS}$$



$f : f^P$

$P_i \rightarrow Q_i$

$E \setminus (P_i \cup Q_i)$ $f_e \rightarrow f_e$

$P_i \cap Q_i$ $f_e \rightarrow f_e$

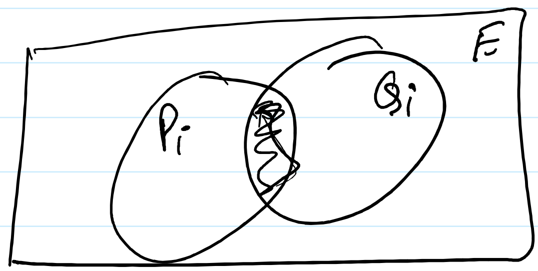
$Q_i \setminus P_i$ $f_e \rightarrow f_e + 1$

$P_i \setminus Q_i$ $f_e \rightarrow f_e - 1$

$$LHS = \sum_{e \in Q_i \setminus P_i} c_e(f_e + 1) + \sum_{e \in Q_i \cap P_i} c_e(f_e) - \sum_{e \in P_i} c_e(f_e)$$

$$LHS = \sum_{e \in Q_i \setminus P_i} c_e(f_e+1) + \sum_{e \in Q_i \cap P_i} c_e(f_e) - \sum_{e \in P_i} c_e(f_e)$$

$$= \underbrace{\sum_{e \in Q_i \setminus P_i} c_e(f_e+1) - \sum_{e \in P_i \setminus Q_i} c_e(f_e)}$$



$$RHS = \phi(Q_i, P_{-i}) - \phi(P_i, P_{-i})$$

$$= \sum_{\substack{e \in E \setminus (Q_i \cup P_i) \\ \cup (Q_i \cap P_i)}} \sum_{k=1}^{f_e} c_e(k) + \underline{\sum_{e \in Q_i \setminus P_i} \sum_{k=1}^{f_e+1} c_e(k)} + \underline{\sum_{e \in P_i \setminus Q_i} \sum_{k=1}^{f_e-1} c_e(k)}$$

$$- \sum_e \sum_{k=1}^{f_e} c_e(k)$$

$$= \sum_{e \in Q_i \setminus P_i} \left(\sum_{k=1}^{f_e+1} c_e(k) \right) - \left(\sum_{k=1}^{f_e} c_e(k) \right) + \sum_{e \in P_i \setminus Q_i} \left(\sum_{k=1}^{f_e-1} c_e(k) \right) - \left(\sum_{k=1}^{f_e} c_e(k) \right)$$

$$= \sum_{e \in Q_i \setminus P_i} c_e(f_e+1) - \sum_{e \in P_i \setminus Q_i} c_e(f_e) = LHS$$

—— X —— X —— X ——

Q: What about PoA?

$$\mathcal{C} = \{ax+b \mid a, b \geq 0\} \text{ affine/linear.}$$

$c_e \in \mathcal{C}$ $\quad c_e(x) = a_e x + b_e$

$c_e(0), c_e(1), c_e(2) \ldots$

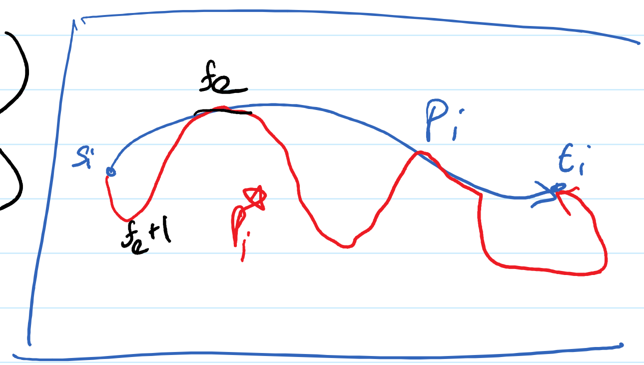
Thm: $\quad PoA(\text{Affine}) \leq \frac{5}{2}. \quad \left(PoA = \frac{\text{cost(worst NE)}}{\text{cost(OPT)}} \right)$

Pf: $\quad P$: NE $\quad \rightarrow \quad f_e$ = # agents taking edge e at P

P^*: OPT $\quad\quad f_e^*$ = " " " " at P^*

$$NE \Rightarrow \forall i: \; c_i(P) \leq c_i(P_i^*, P_{-i})$$

$$= \sum_{e \in P_i^* \setminus P_i} c_e(f_e+1) + \sum_{e \in P_i^* \cap P_i} c_e(f_e)$$



$$\leq \sum_{e \in P_i^*} c_e(f_e+1) \quad (\because c_e \text{ is non-decreasing})$$

$$\Downarrow$$

$$\boxed{\text{Cost}(P) = \sum_i c_i(P) \leq \sum_i c_i(P_i^*, P_{-i})}$$

$$\leq \sum_i \sum_{e \in P_i^*} c_e(f_e+1)$$

$$= \sum_{e \in E} c_e(f_e+1) \sum_{i: e \in P_i^*} 1$$

$\sum_{i: e \in P_i^*} 1$ = # players taking e in P^* $= f_e^*$

$$e\in E \quad \underbrace{\left(i : e \in P_i \right)}_{= f_e^*} \quad \text{in } P$$

$(c_e(x) = a_e x + b_e)$ $$= \sum_{e \in E} \left(a_e (f_e + 1) + b_e \right) \cdot f_e^*$$

$$= \sum_e a_e \underset{x}{\underset{\downarrow}{f_e^*}} \underset{y}{\underset{\downarrow}{(f_e + 1)}} + \sum_e b_e f_e^*$$

Magic $\left(\begin{array}{l} x, y \in \{0, 1, \cdots\} \\ x \cdot (y+1) \le \frac{5}{3} x^2 + \frac{1}{3} y^2 \end{array} \right)$

$$\le \sum_e a_e \left(\frac{5}{3} f_e^{*2} + \frac{1}{3} f_e^2 \right) + \frac{5}{3} \sum_e b_e f_e^*$$

$$+ \frac{1}{3} \sum_e b_e f_e$$

$$= \frac{5}{3} \sum_e a_e f_e^{*2} + b_e f_e^* + \frac{1}{3} \sum_e a_e f_e^2 + b_e f_e$$

$$= \frac{5}{3} \sum_e f_e^* (a_e f_e^* + b_e) + \frac{1}{3} \sum_e f_e (a_e f_e + b_e)$$

$$= \frac{5}{3} \sum_e f_e^* c_e(f_e^*) + \frac{1}{3} \sum_e f_e c_e(f_e)$$

$$= \frac{5}{3} \text{cost}(P^*) + \frac{1}{3} \text{cost}(P)$$

NE

$\Downarrow$ $\Downarrow$

$$\text{cost}(P) \le \boxed{\sum C_i(P_i^*, P_{-i}) \le \frac{5}{3} \text{cost}(P^*) + \frac{1}{3} \text{cost}(P)}$$

$$\mathrm{cost}(P) \le \boxed{\sum_i C_i(P_i^*, P_{-i}) \le \frac{5}{3}\,\mathrm{cost}(P^*) + \frac{1}{3}\,\mathrm{cost}(P)}$$

$$\Rightarrow \left(1-\frac{1}{3}\right)\mathrm{cost}(P) \le \frac{5}{3}\,\mathrm{cost}(P^*)$$

$$\Rightarrow \mathrm{PoA} = \frac{\mathrm{cost}(P)}{\mathrm{cost}(P^*)} \le \frac{5}{2}$$

★ Proof Steps:

① P: NE, P^*: OPT

② NE $\Rightarrow \mathrm{cost}(P) = \sum_i C_i(P) \le \sum_i C_i(P_i^*, P_{-i})$

③ $\sum_i C_i(P_i^*, P_{-i}) \le \underbrace{\frac{5}{3}}_{\lambda}\,\mathrm{cost}(P^*) + \underbrace{\frac{1}{3}}_{\mu}\,\mathrm{cost}(P)$

②③ $\Rightarrow \overbrace{\left(1-\frac{1}{3}\right)}^{(1-\mu)}\mathrm{cost}(P) \le \frac{5}{3}\,\mathrm{cost}(P^*)$

$$\Rightarrow \mathrm{PoA} = \frac{\mathrm{cost}(P)}{\mathrm{cost}(P^*)} = \frac{5/3}{\left(1-\frac{1}{3}\right)} = \frac{\lambda}{1-\mu}$$

★ (λ, μ)-Smooth Games:

$\forall P, P^* \in \mathcal{P}$

$$\sum_i C_i(P_i^*, P_{-i}) \le \lambda\,\mathrm{cost}(P^*) + \mu\,\mathrm{cost}(P)$$

$$\boxed{\sum_i c_i(P_i^*, P_{-i}) \le \lambda \, cost(P^*) + \mu \, cost(P)}$$

$$\Downarrow$$

$$PoA^{CCE} \le \frac{\lambda}{1-\mu}$$

$$PoA^{PNE} \le PoA^{NE} \le PoA^{CE} \le PoA^{CCE} \le \frac{\lambda}{1-\mu}$$