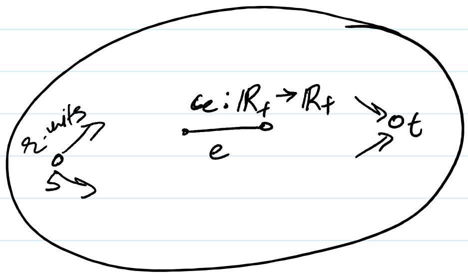
Non-atomic Routing Games ← each individual is infinitesimal :

Tuesday, November 2, 2021 1:54 PM

Recall : Directed graph n/w $G=(V,E)$

c_e : $\underbrace{\text{non-neg, non-decreasing, continuous}}_{\text{valid}}$.

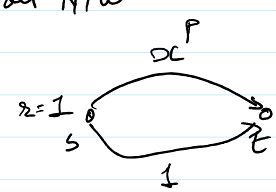


NE: "flow only on least cost paths".

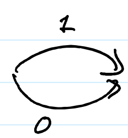
Q: How bad the total cost of the system is at any NE compared to OPT.

$$PoA = \frac{\text{Cost (worst NE)}}{\text{cost (OPT)}}$$

★ Pigou N/W



Only NE :



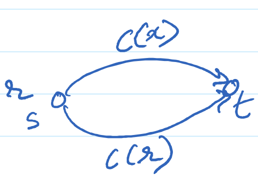
$\lim_{p\to\infty} PoA \to \infty$

Today :

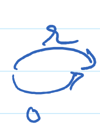
$\mathcal{C}$: Class of "valid" cost functions.

Thm : PoA for any given n/w G w/ costs from $\mathcal{C}$ $\leq$ $\underbrace{\text{PoA of Pigou w/ costs from } \mathcal{C} \cup \{\text{const func}\}}_{\alpha(\mathcal{C})?}$

★ $\alpha(\mathcal{C})$:
$c \in \mathcal{C}$

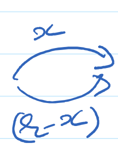


→ NE:

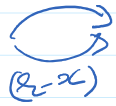


$\text{cost (NE)} = r \cdot c(r)$

OPT :



: $\inf_{0 \le x \le r} \underbrace{x \cdot c(x) + (r-x)\, c(r)}$

OPT :  : $\inf_{0\le x\le r} \; x \cdot c(x) + (r-x)\,c(r)$

$$PoA = \sup_{x\ge 0} \frac{r \cdot c(r)}{c(x) + (r-x)\,c(r)}$$

Claim 1 : $\arg\inf_{x\ge 0} \; \le r$

Pf : $\frac{d}{dx}\left(x\,c(x) + (r-x)\,c(r)\right) = 0$

$\Rightarrow c(x) + x\,c'(x) - c(r) = 0$

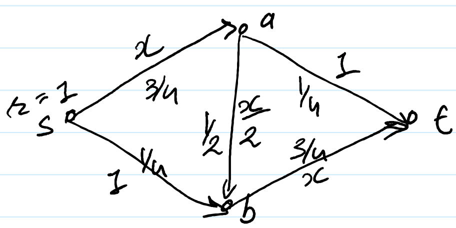
$\Rightarrow c(x) + \underbrace{x\,c'(x)}_{\ge 0} = c(r)$

$\Rightarrow c(x) \le c(r) \Rightarrow x \le r$

$$\alpha(\mathcal{C}) = \sup_{\substack{c\in\mathcal{C},\, r\ge 0 \\ x\ge 0}} \frac{r\, c(r)}{c(x) + (r-x)\,c(r)}$$

Thm: Given $\mathcal{C}$; Given directed n/w $G=(V,E)$, $s,t \in V$, $\forall e: c_e \in \mathcal{C}$; $r \ge 0$ units of flow to be sent from s to t.

$$PoA \le \alpha(\mathcal{C})$$



☆ Notations :

f: flow of r-units from s to t

$\mathcal{P}$: set of all s-t paths

$p\in\mathcal{P}$: f_p: flow on path p

f_e : flow on edge e $= \sum_{P\in\mathcal{P}: e\in P} f_p$

Cost on edge e is $c_e(f_e)$

Cost on path p $= \sum_{e\in P} c_e(f_e) = C_p(f)$

$\mathcal{P} = \{\underset{P_1}{s\text{-}a\text{-}t},\ \underset{P_2}{s\text{-}b\text{-}t},\ \underset{P_3}{s\text{-}a\text{-}b\text{-}t}\}$

$f_{P_1} = \frac{1}{4} = f_{P_2}$, $f_{P_3} = \frac{1}{2}$

$f_{(a,b)} = 1/2$, $f_{(s,a)} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} = f_{(b,t)}$

$f_{(a,t)} = 1/2 = f_{(s,b)}$

$C_{P_1}(f) = 1 + 3/4$

$C_{P_2}(f) = 3/4 + 1$

$C_{P_3}(f) = \frac{3}{4} + \frac{1}{4} + \frac{3}{4}$

$c_{P_3}(f) = \frac{2}{4} + \frac{1}{4} + \frac{1}{4}$

$$\boxed{\begin{array}{l} \text{NE: "flow only on least cost paths"} \\ \forall P \in \mathcal{P} \quad f_P > 0 \Rightarrow c_P(f) = \min_{Q \in \mathcal{P}} c_Q(f) \end{array}}$$

Total $cost(f) = cost(f)$

$$= \sum_{P \in \mathcal{P}} f_P c_P(f) = \sum_{e \in E} f_e c_e(f_e)$$

? Exe.

f: NE flow of r-units

f^*: OPT flow of r-units $= \arg\min_{f'} \ \text{total cost}(f')$

Claim 2: $\sum_{e \in E} (f_e^* - f_e) \, \underline{c_e(f_e)} \geq 0$

Pf: $\boxed{L = \min_{P \in \mathcal{P}} c_P(f)}$, $\quad \sum_{P \in \mathcal{P}} f_P = r = \sum_{P \in \mathcal{P}} f_P^*$

$$cost(f) = \sum_{P \in \mathcal{P}} f_P c_P(f)$$
$$= L \cdot \sum_{P \in \mathcal{P}} f_P \qquad \left(\because \text{ at NE } f_P > 0 \Rightarrow c_P(f) = L \right)$$
$$= L \cdot r$$

$$cost\left(f^* \text{ w/ costs fixed to } c_e(f_e) \text{ const func}\right) = \sum_{P \in \mathcal{P}} f_P^* c_P(f) \geq L \sum_{P \in \mathcal{P}} f_P^*$$
$$= L \cdot r$$
$$= cost(f)$$

$$\sum_{e \in E} f_e^* c_e(f_e) \geq = \sum_{e \in E} f_e c_e(f_e)$$

$$\Rightarrow \sum_{e \in E} (f_e^* - f_e) c_e(f_e) \geq 0$$

f: worst NE flow, f^*: OPT flow

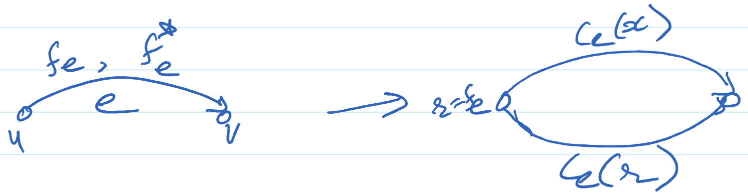
Proof of the Thm:

$$PoA = \frac{\sum_e f_e c_e(f_e)}{\sum f_e^* c_e(f_e^*)} \overset{TP?}{\leq} \alpha(\mathcal{C})$$

$$\mathrm{PoA} = \frac{\quad e}{\sum_e f^*_e\, c_e(f^*_e)} \le \alpha(\mathcal{C})$$

If, $\forall e \in E$ $\quad \dfrac{f_e\, c_e(f_e)}{\alpha(\mathcal{C})} \le f^*_e\, c_e(f^*_e)$ then done

Fix $e \in E$



$c_e \in \mathcal{C}$

$$\alpha(\mathcal{C}) \geq \frac{r \cdot c(r)}{x\, c(x) + (r-x)\, c(r)} \qquad \forall x \geq 0$$

$$\underset{\substack{c = c_e \\ r = f_e \\ x = f^*_e}}{=} \frac{f_e\, c_e(f_e)}{f^*_e\, c_e(f^*_e) + (f_e - f^*_e)\, c_e(f_e)}$$

$$\sum_e f^*_e\, c_e(f^*_e) \geq \underbrace{\sum_e (f^*_e - f_e)\, c_e(f_e)}_{\substack{\geq \\ 0 \ (\because \text{ claim } 1)}} + \sum_e \frac{f_e\, c_e(f_e)}{\alpha(\mathcal{C})}$$

$$\geq \frac{\sum_e f_e\, c_e(f_e)}{\alpha(\mathcal{C})}$$

$$\Rightarrow \mathrm{PoA} = \frac{\sum_e f_e\, c_e(f_e)}{\sum_e f^*_e\, c_e(f^*_e)} \leq \alpha(\mathcal{C})$$

$$\mathcal{C} = \{ax + b \mid a, b \geq 0\} \qquad c(x) = ax + b$$

$$\alpha(\ell) = \sup_{\substack{\boxed{a,b\geq 0,\ r\geq 0} \\ x\geq 0}} \frac{r\cdot(ar+b)}{x(ax+b)+(r-x)(ar+b)}$$

fixing $a, b, r \geq 0$, x s.t. $\underset{x\geq 0}{\text{argmin}} \left(x(ax+b) + (r-x)(ar+b)\right)$ $= ax^2+bx + r(\ ) - x(ar+b)$

$$\Rightarrow \quad \frac{d}{dx} = 0$$

$$2ax + \cancel{b} - ar - \cancel{b} = 0$$

$$2\cancel{a}x = \cancel{a}r \Rightarrow x = r/2$$

$$\alpha(\ell) = \sup_{a,b,r\geq 0} \frac{\cancel{r}\cdot(ar+b)}{\frac{\cancel{r}}{2}\left(\frac{ar}{2}+b\right) + \frac{\cancel{r}}{2}(ar+b)}$$

$$= \ '' \ \frac{ar+b}{\frac{ar}{4}+\frac{ar}{2}+b}$$

$$= \sup_{a,b,r\geq 0} \frac{ar+b}{ar(3/4)+b}$$ b=0 is the best

$$= \sup_{a,r\geq 0} \frac{\cancel{ar}}{\cancel{ar}(3/4)}$$

$$= 4/3$$

N/W Over provisioning:

ISP's generally wants to provide "more capacity than req."

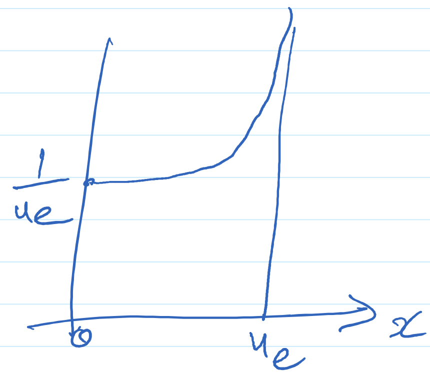
↳ Improves the overall delays

↳ Cheaper than designing + implementing smart QoS.

$$c_e(x) = \frac{1}{u_e - x} \quad \text{if } x \le u_e$$

↑ capacity

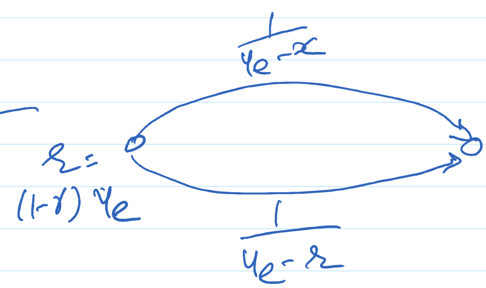
$$= \infty \quad \text{o.w.}$$



* n/w is γ-over provisioned if at NE

$f_e \le (1-\gamma) u_e$ (γ fraction of capacity is unused)

$$PoA \le \frac{1}{2}\left(1 + \frac{1}{\sqrt{\gamma}}\right) \leftarrow$$



$\gamma = 1 \Rightarrow PoA = 1$

$\gamma = 0 \Rightarrow PoA \to \infty$

$\gamma = 0.1 \Rightarrow PoA \sim 2$

Thm: Given $\mathcal{C}$; n/w $G = (V, E)$, s.t. $\forall e \in E$ $c_e \in \mathcal{C}$, $s, t \in V$;

for $r \ge 0$ send r flow from s-t.

f: NE flow of r units

f^*: OPT flow of $2r$ units

$\equiv$ (ISP example: NE flow w/ capacities u_e; OPT flow w/ capacities $\frac{u_e}{2}$)

Then

Then

$\text{cost}(f) \le \text{cost}(f^*)$

Pf:

$$L = \min_{p \in \mathcal{P}} c_p(f)$$

$$\text{Cost}(f) = \sum_p f_p\, c_p(f)$$
$$= L \underbrace{\sum_p f_p}_{=r} \qquad (\because \text{at NE, } f_p > 0 \Rightarrow c_p(f) = L)$$
$$= L \cdot r$$

$$\text{Cost}\left(f^* \text{ w/ costs fixed to } c_e(f_e) \text{ on each } e \in E\right) = \sum_p f_p^* \underbrace{c_p(f)}_{\ge L} \ge L \cdot \underbrace{\sum_p f_p^*}_{= 2r} = 2Lr$$

$$\text{cost}(f) = L \cdot r \le \boxed{\sum_p f_p^* c_p(f) - L \cdot r \;\boxed{\overset{?}{\le}}\; \sum_p f_p^* c_p(f^*)} = \text{Cost}(f^*)$$

then done!

$$\Updownarrow$$

$$\sum_p f_p^* c_p(f) - \sum_p f_p c_p(f) \le \sum_p f_p^* c_p(f^*)$$

$$\Updownarrow$$

$$\sum_e f_e^* c_e(f_e) - \sum_e f_e c_e(f_e) \le \sum_e f_e^* c_e(f_e^*)$$

$$\Uparrow$$

$$\Uparrow$$

$\forall e \in E.$ $f_e^* c_e(f_e) - f_e c_e(f_e) \le f_e^* c_e(f_e^*)$

$$\Updownarrow$$

$$f_e^* \left(c_e(f_e) - c_e(f_e^*) \right) \le f_e c_e(f_e)$$

case I: $f_e \ge f_e^* \Rightarrow f_e^* c_e(f_e) \le f_e c_e(f_e)$

$\Rightarrow f_e^* \left(c_e(f_e) - c_e(f_e^*) \right) \le f_e c_e(f_e)$

case II: $f_e < f_e^* \Rightarrow c_e(f_e) - c_e(f_e^*) \le 0$

$\Rightarrow f_e^* \left(c_e(f_e) - c_e(f_e^*) \right) \le 0 \le f_e c_e(f_e)$ ∎