cs579 Computational Complexity

Problem Set #6

Prof. Michael A. Forbes — Due: *Thu., 2023-11-30 12:00*

All problems are of equal value.

1. (Arora-Barak 12.7) Let $f : \{0,1\}^n \to \{0,1\}$ be a boolean function. Recall that the *degree* of f over a field $\mathbb{F}$ (denoted $\deg_{\mathbb{F}} f$) is the minimum degree of a polynomial $p \in \mathbb{F}[x_1, \ldots, x_n]$ such that $f(x) = p(x)$ for all $x \in \{0,1\}^n$. Show that for any field $\mathbb{F}$, $\deg_{\mathbb{F}} f \le D(f)$, where $D(f)$ is the deterministic decision-tree complexity of f.

2. (Normal Form for Formulas) Given an unbounded fan-in $\{\text{AND}, \text{OR}, \text{NOT}\}$-formula of size-$s$, where size here is the number of $\{\text{AND}, \text{OR}\}$-gates, show that there is an equivalent formula of size $s' \le s$ where all negations occur at the bottom of the formula, and all $\{\text{AND}, \text{OR}\}$-gates have fan-in ≥ 2. Show that s' is bounded by the number of leaves of the resulting formula.

3. Let $\ell : \{0,1\}^\star \to \mathbb{N}$ be a *length* function, meaning that $\ell(x)$ is computable in $\mathsf{poly}(|x|)$ time and $\mathsf{poly}(|x|) \le \ell(x) \le \mathsf{poly}(|x|)$. A function $f : \{0,1\}^\star \to \{0,1\}^\star$ is *downward self-reducible* with respect to ℓ if

   - If $\ell(x) = 0$ then $f(x)$ is computable in $\mathsf{poly}(|x|)$ time.
   - In general, x can be computed in $\mathsf{poly}(|x|)$ time given oracle access to f on inputs $\{y : \ell(y) < \ell(x)\}$.

   (a) Prove that, under a suitable encoding, $\mathsf{3SAT}$ is downward self-reducible with respect to $\ell(\varphi)$ being the number of variables in φ.

   (b) Show that, under a suitable encoding, computing the number of perfect matchings of a graph is downward self-reducible with respect to some natural length function.

   (c) (Arora-Barak Problem 8.9) Any downward self-reducible function is computable in $\mathsf{poly}(|x|)$ space (ie, PSPACE when f is a language).

4. (Arora-Barak 13.3) Prove that a single tape Turing machine (one whose input tape is also its read-write tape), takes $\Omega(n^2)$ time to decide the palindrome language $\mathsf{PAL} = \{x : x = x^R \in \{0,1\}^\star\}$, where x^R is the reverse of the string x.