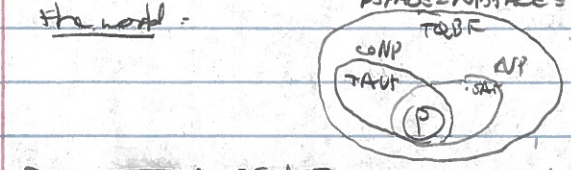


CSE 79 Computational Complexity: Lecture 8 (2023-09-14)

logistics: - pset 2 due 09-21

last lecture: LADDER, FA & NPSPACE [Cook to Levin] [Savitch]
 $NPSPACE(s) = SPACE(s^2)$ [start] → [mid] → [end], reuse space [NPSPACE = PSPACE, vs P ≠ NP]
 TQBF PSPACE-complete [Savitch version of Cook-Levin]

today: space complexity beyond TQBF
 $PSPACE = NPSPACE = coNPSPACE$

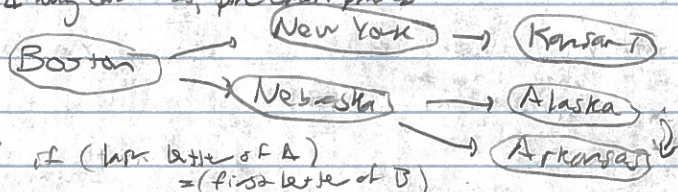


[NP-complete problem abundance]

Q: ~~PSPACE~~ PSPACE complete problems?

A: games

eg: geography [long car rides, pre-smart phone]



- rules:
- $A \rightarrow B$ if (last letter of A) = (first letter of B)
 - two players, player 1 starts
 - players alternate picking edges to follow, no vertex repeats
 - whoever is stuck loses

def: Generalized Geography is this game played on a directed graph $G=(V,E)$

w/ designated start node $s \in V$

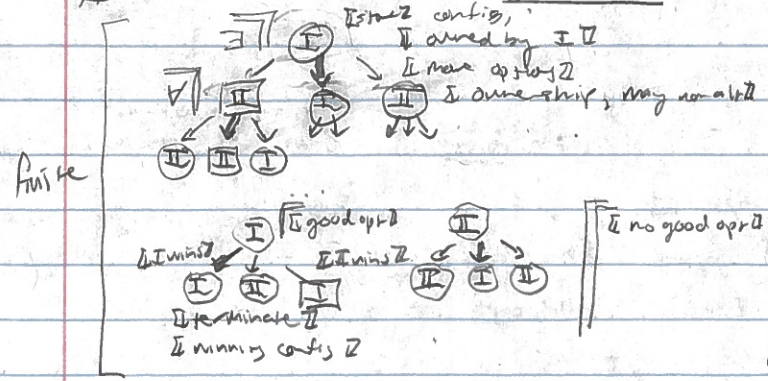
$G, s = \langle G, s \rangle$: player I has winning strategy in G starting at s

Q: well defined?

player I wins if they play perfectly [even if player II plays poorly]

thm: any finite two-player game, w/ no ties, exactly one player has winning strategy

pf: \Rightarrow finite game tree



- terminal configs are wins to I or II
- inductively define optimal play, for each node in tree
- player I wins if node is I child where player I wins
- player I wins if node is II child where player I wins
- player I wins game if player I wins some node

thm: GG is PSPACE complete

pf: - $GG \in PSPACE$: idea: recursion, saving space

\equiv same as in $T \subseteq PSPACE$

$TQBF \leq_p GG$

def: Γ is a game

even less fun? in a boolean formula $\Psi(x_1, \dots, x_n)$

- player I starts
- players alternate choosing $x_i, x_{i+1} \in \{0,1\}$ Γ is odd
- at end, $\Psi(x) = 1 \Rightarrow$ player I wins

lem: player I has winning strategy in formula game on Ψ

iff $\exists x_1, \forall x_2, \dots, \exists x_n \Psi \in TQBF$

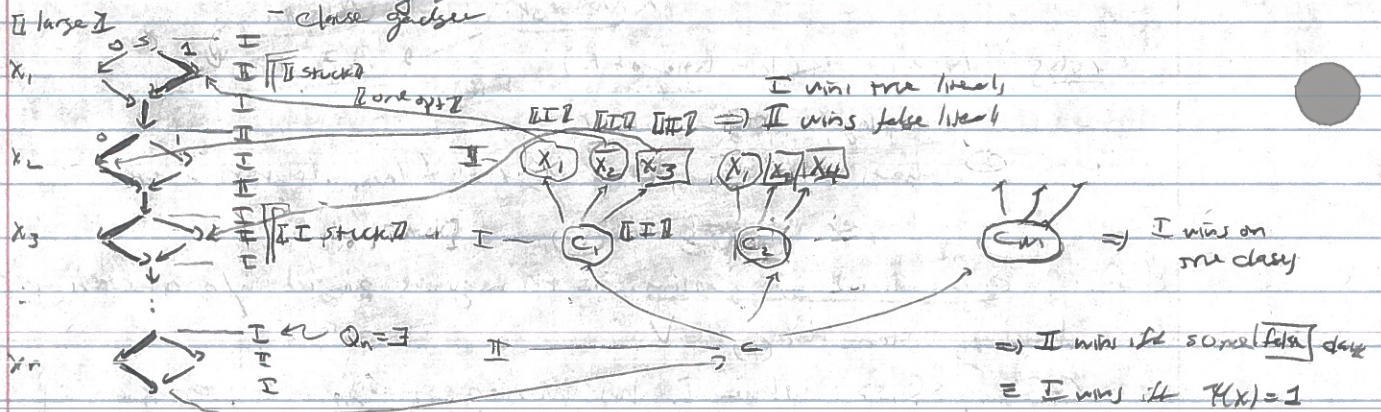
sketch: player I nodes $\exists \exists$ player I wins if true

prop: FORMULA-GAME $\in_p GG$

sketch $\Psi = \exists x_1, \forall x_2, \dots, \exists x_n \Psi$

log: - quantifiers alternate, start and end w/ \exists
 - $\Psi \text{ CNF } [pset \cup \{ \}]$ add dummy vars

work: $G = (V, E)$ directed, $5 \in V$
 - variable gadget
 - clause gadget



prop: $\Psi \in TQBF$ iff $\langle G, s \rangle \in GG$ (by construction)

propagate back to s

complexity: $2G, 57$ physical copyable (clear)

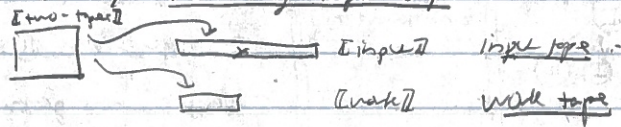
rmk: most "properly generalized" games are PSPACE-hard to decide unless $= P$

Q: sublinear space?
 $o(n)$

A: $\Rightarrow s(n) \geq n$ (always)

Γ trivial objection
 Γ we can be smarter

def: a TM w/ read only input tape



- read only Γ transition function cannot write
- init to w, x, w Γ bounding
- read/write
- work tape: - init to blank

computes in space $s(n)$ if - always halts

- all $x \in \Sigma^+$, $|w| \leq s(n)$ (work) tape cells Γ avoids trivial objection Γ input string

rmk: input often too big in real life || internet ||
 def - $L = \text{SPACE}(lg n)$ is logspace || hidden constant || || natural - pointers - carries ||

|| log || NL = NSPACE($lg n$) is non-deterministic logspace
 prop - $A = \{x : x \in \{0,1\}^* ; x = x^R\}$ || palindrome || $\in L$

pf: algo - "on input x :"
 - for $1 \leq i \leq n = |x|$
 - if $x_i \neq x_{n-i}$, reject
 - accept "

correctness: clear || space ||
 complexity: computing n $O(lg n)$ space || carry ||
 iterating over i

computing x_i - " || carry ||
 - on input tape, move i cells from first symbol
 - copy x_i onto work-tape
 - reset input tape head [to first symbol] } $O(lg n)$ space

computing x_{n-i} - "
 - move to end of input tape
 - move i cells left on input tape
 - copy x_{n-i} onto work tape
 - reset input tape head [to first symbol] } $O(lg n)$ space

|| log NL || \rightarrow total $O(lg n)$ space ||

def - PATH = $\{ \langle G, s, t \rangle = s \rightsquigarrow t \text{ path in directed graph } G ? \}$

prop: PATH \in NL

pf: || same as LADDERDEF ||

algo - " on input $\langle G, s, t \rangle$:"

- $v \leftarrow s$
 - for $\leq |V|$ steps || wlog cycle-free path ||
 - if $v = t$, accept
 - guess neighbor u of v || guess u , check if $uv \in E$ ||
 - $v \leftarrow u$
 - reject "

$O(lg |V|)$ || pointer ||
 $O(lg |V|)$ || carries ||
 $O(lg |V|)$ || adj matrix: carries ||

correctness: clear || LADDERDEF ||

complexity - $O(lg |V|)$

Q: PATH \in L? L vs NL? || evidence? ||

thm [Savitch 7]: $s(n) \geq lg n$ $NSPACE(s) \subseteq SPACE(s^2)$ || same proof, in new model ||

cor - $NL \subseteq SPACE(lg^2 s) \subseteq TIME(n^{O(lg n)})$

prop. $s(n) \geq \lg n$. $SPACE(s) \in TIME(2^{O(s)})$ \parallel same proof \parallel

cor. $L \in P$

prop. $s(n) \geq \lg n$, weakly space constructible. $NSPACE(s) \in TIME(2^{O(s)})$

cor. $NLEP$

\cdot $\lg n$ space constructible

pf. def. config of TM \forall read-only input tape w - current state q new models \parallel no input tape \parallel
 - [work] tape contents \parallel input \parallel
 - head locations \parallel \cdot w \parallel \cdot s \parallel \cdot s \parallel
 \parallel input, markers \parallel

lem: space $s(n) \Rightarrow \# \text{ configs} \in |Q| \cdot |\Gamma|^s \cdot (n+2) \cdot s \leq 2^{O(s)}$, $|C| \leq 2^{O(s)}$, $|C| \leq 2^{O(s)}$, $|C| \leq 2^{O(s)}$

def. the config graph of space- $s(n)$ NTM M on x is directed graph $G = (V, E)$ \parallel \cdot $V = \{ \text{all space } s(n) \text{ configs of } M \}$ \parallel \cdot $E = \{ c \rightarrow c' \text{ in } M \text{ on } x \}$ \parallel \cdot yield \parallel

clm: $M \text{ acc } x$ iff $C_{\text{init}} \rightsquigarrow C_{\text{acc}}$ in G \parallel \cdot log uniq \parallel \parallel \cdot is in SVL \parallel

algo. " on input x :

- construct G
- solve $\langle G, C_{\text{init}}, C_{\text{acc}} \rangle \in \text{PATH}$

poly $|G|$ time
 poly $|G|$ time
 via BFS
 given s
 \parallel space constructible \parallel

correctness - clear

complexity. $|G| \in 2^{O(s)}$

constructing $1^{s(n)}$ in $2^{O(s)}$ time via space constructibility

today. - more PSPACE completeness

- games \parallel unique winner \parallel \parallel \cdot decide \parallel \parallel \cdot game \parallel \parallel \cdot game \parallel \parallel \cdot game \parallel \parallel \cdot game \parallel \parallel
- TQBF \parallel \cdot game \parallel \parallel \cdot game \parallel \parallel \cdot game \parallel \parallel \cdot game \parallel \parallel
- logarithmic space \parallel read-only work tape \parallel
- PALINDROME \parallel \cdot count , point \parallel
- PATH \in NL \parallel \cdot game \parallel
- NL \in P \parallel \cdot SVL \parallel \parallel \cdot SVL \parallel \parallel

next lecture. L vs NL vs coNL

logistics. - psr-2 due 09-21