CS576 Topics in Automated Deduction

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Slides based in part on slides by Tobias Nipkow

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Contact Information

- Office: 2112 SC
- Office Hours:
  - Fridays 11:00am - 12:15pm
  - Also by appointment
  - May add more if desirable
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- Newsgroup: https://campuswire.com/c/GA8DC6DE7/feed
- No TA this semester

Course Structure

- Recommended Texts:
  - Programming and Proving in Isabelle/HOL
    by Tobias Nipkow
  - Isabelle/HOL: A Proof Assistant for Higher-Order Logic
    by Tobias Nipkow, Lawrence C. Paulson, Markus Wenzel
  - Concrete Semantics with Isabelle/HOL
    Tobias Nipkow and Gerwin Klein,
    http://www.concrete-semantics.org
- Credit:
  - Homework (submitted via PrairieLearn) 33%
  - Project and presentation 67%
- No Final Exam

Some Useful Links

- Website for class:
  http://courses.engr.illinois.edu/cs576/sp2026/
- Website for Isabelle:
  http://www.cl.cam.ac.uk/Research/HVG/Isabelle/
- Isabelle mailing list – to join, send mail to:
  isabelle-users@cl.cam.ac.uk

Your Work

- Homework:
  - (Mostly) fairly short exercises in Isabelle
  - Submitted via svn
- Project:
  - Develop a model of a system in Isabelle
  - Prove some substantive properties of model
  - Discuss progress weekly in class
  - Give 20 minute presentation of work at end of course

Course Objectives

- To learn to do formal reasoning
- To learn to model complex problems from computer science
- To learn to given fully rigorous proofs of properties

System Architecture

Isabelle/jEdit	jEdit based interface
Isar	Isabelle proof scripting language
Isabelle/HOL	Isabelle instance for HOL
Isabelle	generic theorem prover
Standard ML	implementation language

jEdit Input

Input of math symbols in jEdit

- via "standard" ascii name: &, |, -->, ...
- via ascii encoding (similar to LaTeX): \<and>, \<or>, ...
- via menu ("Symbols")

Symbol Translations

symbol	$\forall$	$\exists$	λ	$\neg$	$\wedge$
ascii (1)	\<forall>	\<exists>	\<lambda>	\<not>	\<and>
ascii (2)	ALL	EX	%	~	&

symbol	$\vee$	$\longrightarrow$	$\Rightarrow$
ascii (1)	\<or>	\<longrightarrow>	\<Rightarrow>
ascii (2)	|	-->	=>

See Appendix A of tutorial for more complete list

Time for a demo of types and terms
(and a simple lemma)

Overview of Isabelle/HOL

HOL

- HOL = Higher-Order Logic
- HOL = Types + Lambda Calculus + Logic
- HOL has
  - datatypes
  - recursive functions
  - logical operators ($\wedge$, $\vee$, $\longrightarrow$, $\forall$, $\exists$, ...)
- HOL is very similar to a functional programming language
- Higher-order = functions are values, too!

Formulae (Approximation)

- Syntax (in decreasing priority):

$$\begin{array}{lcll} \mathit{form} & ::= & (\mathit{form}) & \mid \ \mathit{term} = \mathit{term} \\ & \mid & \neg \mathit{form} & \mid \ \mathit{form} \wedge \mathit{form} \\ & \mid & \mathit{form} \vee \mathit{form} & \mid \ \mathit{form} \longrightarrow \mathit{form} \\ & \mid & \forall x.\ \mathit{form} & \mid \ \exists x.\ \mathit{form} \end{array}$$

and some others

- Scope of quantifiers: as for to right as possible

Examples

- $\neg A \wedge B \vee C \equiv ((\neg A) \wedge B) \vee C$
- $A \wedge B = C \equiv A \wedge (B = C)$
- $\forall x.\ P\ x \wedge Q\ x \equiv \forall x.\ (P\ x \wedge Q\ x)$
- $\forall x. \exists y.\ P\ x\ y \wedge Q\ x \equiv \forall x. (\exists y.\ (P\ x\ y \wedge Q\ x))$

Formulae

- Abbreviations:
  $\forall x\ y.\ P\ x\ y \equiv \forall x. \forall y.\ P\ x\ y \quad (\forall, \exists, \lambda, \ldots)$
- Hiding and renaming:
  $\forall x\ y.\ (\forall x.\ P\ x\ y) \wedge Q\ x\ y \equiv \forall x_0\ y. (\forall x_1. P\ x_1\ y) \wedge Q\ x_0\ y$
- Parentheses:
  - $\wedge$, $\vee$, and $\longrightarrow$ associate to the right:
    $A \wedge B \wedge C \equiv A \wedge (B \wedge C)$
  - $A \longrightarrow B \longrightarrow C \ \equiv A \longrightarrow (B \longrightarrow C)$
    $\not\equiv (A \longrightarrow B) \longrightarrow C$!

Warning!

Quantifiers have low priority (broad scope) and may need to be parenthesized:

$$! \quad \forall x.\ P\ x \wedge Q\ x \not\equiv (\forall x.\ Px) \wedge Q\ x \quad !$$

Types

Syntax:

τ	$::=$	(τ)	
	$\mid$	bool | nat | ...	base types
	$\mid$	'a | 'b | ...	type variables
	$\mid$	$\tau \Rightarrow \tau$	total functions (ascii : =>)
	$\mid$	$\tau \times \tau$	pairs (ascii : *)
	$\mid$	τ list	lists
	$\mid$	...	user-defined types

Parentheses: $T1 \Rightarrow T2 \Rightarrow T3 \equiv T1 \Rightarrow (T2 \Rightarrow T3)$

Terms: Basic syntax

Syntax:

term	$::=$	(*term*)	
	$\mid$	*c* | *x*	constant or variable (identifier)
	$\mid$	*term term*	function application
	$\mid$	$\lambda x.$ *term*	function "abstraction"
	$\mid$	...	lots of syntactic sugar

Examples: $f\ (g\ x)\ y \quad h\ (\lambda x.\ f\ (g\ x))$
Parentheses: $f\ a_1\ a_2\ a_3 \equiv ((f\ a_1)\ a_2)\ a_3$
Note: Formulae are terms

λ-calculus in a nutshell

Informal notation: $t[x]$
term t with 0 or more free occurrences of x

- **Function application:**
  $f\ a$ is the function f called with argument a.
- **Function abstraction:**
  $\lambda x.t[x]$ is the function with formal parameter x and body/result $t[x]$, i.e. $x \mapsto t[x]$.

λ-calculus in a nutshell

- **Computation:**
  Replace formal parameter by actual value
  ("β-reduction"): $(\lambda x.t[x])a \rightsquigarrow_\beta t[a]$

Example: $(\lambda x.\ x + 5)\ 3 \rightsquigarrow_\beta (3 + 5)$

Isabelle performs β-reduction automatically

Isabelle considers $(\lambda x.t[x])a$ and $t[a]$ equivalent

Terms and Types

Terms must be well-typed!

The argument of every function call must be of the right type

Notation: $t :: \tau$ means t is well-typed term of type τ

Type Inference

- Isabelle automatically computes ("infers") the type of each variable in a term.
- In the presence of *overloaded* functions (functions with multiple, unrelated types) not always possible.
- User can help with type annotations inside the term.
- **Example:** $\mathrm{f(x::nat)}$

Currying

- **Curried:** $\mathrm{f} :: \tau_1 \Rightarrow \tau_2 \Rightarrow \tau$
- **Tupled:** $\mathrm{f} :: \tau_1 \times \tau_2 \Rightarrow \tau$

Advantage: partial application $\mathrm{f\ a_1}$ with $a_1 :: \tau$
Moral: Thou shalt curry your functions (most of the time :-)).

Terms: Syntactic Sugar

Some predefined syntactic sugar:

- Infix: $+$, $-$, $\#$, @, ...
- Mixfix: $\mathrm{if\_then\_else\_}$, $\mathrm{case\_of\_}$, ...
- Binders: $\forall \mathrm{x.P\ x}$ means $(\forall)(\lambda x.\ P\ x)$

Prefix binds more strongly than infix:

$$!\quad \mathrm{f\ x + y \equiv (f\ x) + y \not\equiv f\ (x + y)} \quad !$$

Type bool

Formulae = terms of type bool

True::bool

False::bool

$\neg$ :: bool $\Rightarrow$ bool

$\wedge$, $\vee$, ... :: bool $\Rightarrow$ bool $\Rightarrow$ bool

$\vdots$

if-and-only-if: $=$ but binds more tightly

Type nat

0::nat

Suc :: nat $\Rightarrow$ nat

$+$, $\times$, ... :: nat $\Rightarrow$ nat $\Rightarrow$ nat

$\vdots$

Overloading

! Numbers and arithmetic operations are overloaded:

0, 1, 2, ... :: nat or real (or others)

$+ :: nat \Rightarrow nat \Rightarrow nat$ and

$+ :: real \Rightarrow real \Rightarrow real$ (and others)

You need type annotations: $1 :: nat$, $x + (y :: nat)$

...unless the context is unambiguous: Suc 0

Type `list`

- `[ ]`: empty list
- x # xs: list with first element x ("head") and rest xs ("tail")
- Syntactic sugar: $[x_1, \ldots, x_n] \equiv x_1 \# \ldots \# x_n \# [\,]$

List is supported be a large library:
hd, tl, map, size, filter, set, nth, take, drop, distinct, ...

Don't reinvent, reuse!
$\rightsquigarrow$ `HOL/List.thy`

A Recursive datatype

```
datatype 'a list = Nil    ("[]")
                 | Cons 'a "'a list"  (infixr "#'' 65)
```

`[]`: empty list

`x # xs`: list with head x::'a, tail xs::'a list

A toy list: False # (True # [ ])

Syntactic sugar: [False, True]

Concrete Syntax

When writing terms and types in `.thy` files

Types and terms need to be enclosed in `"..."`

Except for single identifiers, e.g. `'a`

`" ..."` won't always be shown on slides

Structural Induction on Lists

P xs holds for all lists xs if

- P [ ]
- and for arbitrary y and ys, P ys implies P (y # ys)

$$\frac{\begin{array}{c}\texttt{P ys}\\ \vdots \\ \texttt{P (y \# ys)}\end{array}}{\texttt{P xs}}$$

A Recursive Function: List Append

Definition by *primitive recursion*:

```
primrec app :: "'a list ⇒ 'a list ⇒ 'a list
where
app [ ] ys = ____
app (x # xs) ys = ____app xs ...____
```

One rule per constructor
Recursive calls only applied to constructor arguments
Guarantees termination (total function)

Demo: Append and Reverse