Random Walks

Ex 0 1D walk with barrier

- Ant starts at position $i$.
- At each step,
  - if $0 < i < n$, move to $i-1$ w. prob $\frac{1}{2}$
  - if $i = 0$, stop.
  - if $i = n$, move to $n-1$ w. prob $\frac{1}{2}$

What is expected at steps?

Note - without barrier at $n$, this is "Gambler's ruin"
- (with prob 1, will reach position 0
  - but $E(\text{all steps to reach 0}) = \infty$)

Graph representation:

Markov chain

Sol'n:
- Let $t_i$: expected # steps to reach 0
  - starting at $i$.

Then

- $t_0 = 0$
- $t_i = 1 + \frac{1}{2} t_{i-1} + \frac{1}{2} t_{i+1}$ for $0 < i < n$
\[ t_0 = 0 \]
\[ t_c = 1 + \frac{1}{2} t_{c+1} + \frac{1}{2} t_{c+1} \quad \text{left} \]
\[ t_n = 1 + t_{n-1} \quad \text{right} \]

\[ \Rightarrow t_i + t_i = 2 + t_{i+1} + t_{i+1} \]
\[ t_i - t_{i+1} = 2 + t_{i+1} - t_i \]

Let \( d_i = t_i - t_{i+1} \)

\[ \Rightarrow \begin{cases} d_i = 2 + d_{i+1} \\ d_n = 1 \end{cases} \]

\[ \Rightarrow d_{n-1} = 3, \quad d_{n-2} = 5, \quad d_{n-3} = 7, \ldots \]
\[ d_i = 2(n-i) + 1 \]

\[ \Rightarrow \begin{cases} t_i = t_{i+1} + 2(n-i)+1 \\ t_0 = 0 \end{cases} \]

\[ \Rightarrow t_i = \sum_{j=1}^{i} (2(n-j)+1) \]
\[ = \sum_{j=1}^{i} (2n+1 - 2j) \]
\[ = (2n+1)i - 2 \sum_{j=1}^{i} j \]
\[ = 2ni - i^2 \]
\[ \leq \left( \frac{n^2}{2} \right) \quad \text{quadratic!} \]

\[ \text{Satisfiability (SAT)} \]
\[ \text{in CNF (conjunctive normal form)} \]
Satisfiability (SAT)

Given Boolean expr $E$ in CNF (conjunctive normal-form)

with $n$ vars, $m$ clauses, Booleans
does there exist assignment of values to vars st.
$E$ evaluates to true

\[ E = (x_1 \lor x_2 \lor x_3) \land (x_1 \lor \overline{x}_2) \land (x_1 \lor \overline{x}_2 \lor \overline{x}_3) \land (x_2 \lor \overline{x}_3) \]

\[ \text{Yes } x_1 = 1, x_2 = 0, x_3 = 0. \]

brute force $O^*(2^n)$-time ($O^*$ hides polynomial factors)

NP-complete, so not likely polytime

but can beat $2^n$ ...

\text{e.g. 3SAT:} \quad \begin{align*}
\text{take a clause } & x_1 \lor x_2 \lor x_3, \\
\text{branch on 7 cases} & \quad \Rightarrow T(n) = 7T(n-3) + O(m) \\
\Rightarrow O^*(7^{n/3}) = O^*(1.92^n). \\
\end{align*}

'85: $O^*(1.618^n)$

'93: $O^*(1.579^n)$

\vdots

'96: $O^*(1.476^n)$

\vdots

Papadimitriou's Rand. Alg'n ('91)

\begin{itemize}
\item start with any assignment $A$
\item repeat $t$ times:
  \begin{itemize}
  \item if $A$ satisfies $E$ then return "yes"
  \item pick any clause $C$ not satisfied by $A$
  \item flip any literal $x$ in $C$ at random
  \end{itemize}
\end{itemize}
if \( A \) satisfies \\
   - pick any clause \( C \) not satisfied by \( A \) \\
   - choose one of its literals \( \alpha \) in \( C \) at random \\
   - flip \( \alpha \)'s value in \( A \) \\
}

return "No"

(1-sided Monte Carlo)

\[ C_i \cap C_2 \cap C_3 \cap C_4 \]

\[ (x_1 \lor x_2 \lor x_3) \land (x_1 \lor x_2) \land (\overline{x}_1 \lor x_2 \lor x_3) \land (x_2 \lor x_3) \]

Start with \( x_1 = 1, x_2 = 0, x_3 = 1 \).

\( C_4 \) not sat, flip \( x_2 \): \( x_1 = 1, x_2 = 1, x_3 = 1 \).

\( C_3 \) not sat, flip \( x_1 \): \( x_1 = 0, x_2 = 1, x_3 = 1 \).

\( C_2 \) not sat: flip \( x_2 \): \( x_1 = 0, x_2 = 0, x_3 = 1 \).

\[ \vdots \]

---

**Analysis for 2SAT:**

If \( E \) not sat, correct always.

Say \( E \) satisfied by some assignment \( A^* \).

Let \( i = \) Hamming dist. between \( A \) and \( A^* \).

\( (\# \) vars assigned diff. values by \( A \) & \( A^* \))

Suppose clause \( C = \overline{\alpha_j} \lor \overline{\alpha_k} \) is picked.

In \( A \), both \( \alpha_j, \alpha_k \) are 0.

In \( A^* \), at least one of \( \alpha_j, \alpha_k \) is 1.

w.l.o.g. \( \alpha_j = 1 \), say.

w. prob \( \frac{1}{2} \), flip \( \alpha_j \) \( \Rightarrow \) \( i \) decrements

w. prob \( \frac{1}{2} \), flip \( \alpha_k \) \( \Rightarrow \) \( i \) increments or \( i \) decrements

\[ 0 \leq i \leq n \] always.

If \( i = 0 \), stop.

By Ex 0, expected \# steps \( \leq n^2 \).

\( \text{Set } t = \overline{2n^2} \) by Markov's Ineq. \( \Pr(\text{err}) < \frac{1}{2} \).
Set $t = 2^n \Rightarrow \Pr(\text{err}) \leq 1$.

Poly time for 2-SAT!

(not new, for 2-SAT)

What about 3-SAT?

$$t_i = 1 + \frac{1}{3} t_{i-1} + \frac{2}{3} t_{i+1}$$

$$t_i = 1 + t_{i-1} + 2t_{i+1}$$

$$t_i = t_{i-1} + 3 + 2(t_{i+1} - t_i)$$

$$d_i = 3 + 2d_{i+1}$$

$$\Rightarrow \Theta(2^n) \quad \text{bad!!}$$

Schönig's Alg'm ('99)

Start with a random assignment $A$

Do same as Papadimitriou

But for $t = 3n$ steps

Claim $\Pr(\text{correct}) \geq \left(\frac{3}{4}\right)^n$ for 3-SAT.

Repeat $c(\frac{4}{3})^n$ times

$\Rightarrow \Pr(\text{err}) \leq \left(1 - \left(\frac{3}{4}\right)^n\right)^c \leq e^{-c}$

$\Rightarrow O^c \left(\left(\frac{4}{3}\right)^n\right) = O^c(1.33...^n)$