CS 574, Spring 2021
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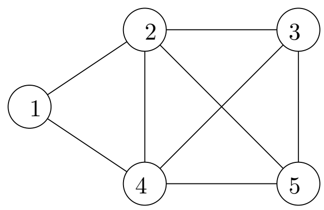
Homework 5 (due April 30 Friday 5pm (CT))

Note: This homework is shorter and will be weighted less (counted as half of a regular homework).

Instructions: You may work in groups of at most 3; submit one set of solutions per group. Always acknowledge any discussions you have with other people and any sources you have used (although most homework problems should be doable without using outside sources). In any case, *solutions must be written entirely in your own words.*

1. [*18 pts*] Consider the Markov chain associated with the following undirected graph:



   (a) [*6 pts*] Let π be its stationary distribution. Write down a linear system of equations for $\pi = (\pi_1, \ldots, \pi_5)$ and solve for $\pi_1, \ldots, \pi_5$. (The calculations here should be doable by hand.)

   (b) [*6 pts*] Recall that h_{j1} denotes the expected number of steps to reach state 1 starting at state j. Note that for $j = 1$, we take h_{11} to be the expected number of steps to return to state 1 starting at state 1. Write down a linear system of equations for $h_{11}, h_{21}, \ldots, h_{51}$ (in a manner similar to the one-dimensional random walk example from class) and solve for $h_{11}, h_{21}, \ldots, h_{51}$.

   (c) [*6 pts*] In general, for any Markov chain where a stationary distribution π exists with $\pi_i > 0$, give a direct proof that $h_{11} = 1/\pi_1$. Do not invoke the Fundamental Theorem, but rather use the linear systems of equations for the π_i's and the h_{j1}'s.

2. [*32 pts*] Consider an undirected connected graph $G = (V, E)$. As in class, let h_{uv} be the expected number of steps to reach v in a random walk starting at u (the hitting time), and let C_s be the expected number of steps to visit all vertices in a random walk starting at s (the cover time). Let $h^* = \max_{u,v \in V} h_{uv}$ and $C^* = \max_{s \in V} C_s$.

   A straightforward argument (by summing h_{uv} over the edges of a spanning tree) implies that $C^* \leq O(n) \cdot h^*$ (actually, it shows $C^* \leq O(n) \cdot \max_{u,v \in V:\, uv \in E} h_{uv}$). You will consider two approaches to prove the stronger bound $C^* \leq O(\log n) \cdot h^*$.

   First approach:

   (a) [*4 pts*] For any fixed vertices u and v, first prove that the probability that a random walk of length $\lceil 2h^* \rceil$ starting at u does not visit v is at most $1/2$.

(b) [*6 pts*] For any fixed vertices s and v, prove that the probability that a random walk of length $\lceil 2h^* \rceil \cdot \lceil c \log_2 n \rceil$ starting at s does not visit v is at most $1/n^c$ (by using (a)).

(c) [*6 pts*] Conclude that $C^* \leq O(h^* \log n)$ (by using (b)).

Second approach:

(d) [*4 pts*] Consider a fixed (not random) walk starting at u. Consider a random permutation $v_1, \ldots, v_n$ of V. Let T_i be the first time when v_i is visited in the walk. Let $M_i = \max(T_1, \ldots, T_i)$. Show that $\Pr[M_i \neq M_{i-1}] = 1/i$.

(e) [*6 pts*] Now consider a fixed permutation $v_1, \ldots, v_n$ of V. Consider a random walk starting at u. Let T_i be the first time when v_i is visited in the walk. Let $M_i = \max(T_1, \ldots, T_i)$. Show that $E[M_i - M_{i-1} \mid M_i \neq M_{i-1}] \leq h^*$.

Hint: argue that the event $M_i \neq M_{i-1}$ is dependent only on the part of the walk before time M_{i-1}.

(f) [*6 pts*] Conclude that $C^* \leq O(h^* \log n)$ (by using (d) and (e)). Is the hidden constant factor here better or worse than in (c)?