Chapter 29

Random Walks I

By Sariel Har-Peled, March 21, 2024[①]

> "A drunk man will find his way home; a drunk bird may wander forever."
>
> Anonymous,

29.1. Definitions

Let $\mathsf{G} = \mathsf{G}(\mathsf{V}, \mathsf{E})$ be an undirected connected graph. For $v \in \mathsf{V}$, let $\Gamma(v)$ denote the set of neighbors of v in G; that is, $\Gamma(v) = \left\{ u \,\middle|\, vu \in \mathsf{E}(\mathsf{G}) \right\}$. A ***random walk*** on G is the following process: Starting from a vertex v_0, we randomly choose one of the neighbors of v_0, and set it to be v_1. We continue in this fashion, in the ith step choosing v_i, such that $v_i \in \Gamma(v_{i-1})$. It would be interesting to investigate the random walk process. Questions of interest include:

(A) How long does it take to arrive from a vertex v to a vertex u in G?
(B) How long does it take to visit all the vertices in the graph.
(C) If we start from an arbitrary vertex v_0, how long the random walk has to be such that the location of the random walk in the ith step is uniformly (or near uniformly) distributed on $\mathsf{V}(\mathsf{G})$?

Example 29.1.1. In the complete graph K_n, visiting all the vertices takes in expectation $O(n \log n)$ time, as this is the coupon collector problem with $n-1$ coupons. Indeed, the probability we did not visit a specific vertex v by the ith step of the random walk is $\leq (1 - 1/n)^{i-1} \leq e^{-(i-1)/n} \leq 1/n^{10}$, for $i = \Omega(n \log n)$. As such, with high probability, the random walk visited all the vertex of K_n. Similarly, arriving from u to v, takes in expectation $n-1$ steps of a random walk, as the probability of visiting v at every step of the walk is $p = 1/(n-1)$, and the length of the walk till we visit v is a geometric random variable with expectation $1/p$.

29.1.1. Walking on grids and lines

Lemma 29.1.2 (Stirling's formula). *For any integer $n \geq 1$, it holds $n! \approx \sqrt{2\pi n}\,(n/e)^n$.*

29.1.1.1. Walking on the line

Lemma 29.1.3. *Consider the infinite random walk on the integer line, starting from 0. Here, the vertices are the integer numbers, and from a vertex k, one walks with probability $1/2$ either to $k-1$ or $k+1$. The expected number of times that such a walk visits 0 is unbounded.*

Proof: The probability that in the $2i$th step we visit 0 is $\frac{1}{2^{2i}}\binom{2i}{i}$, As such, the expected number of times we visit the origin is

$$\sum_{i=1}^{\infty} \frac{1}{2^{2i}} \binom{2i}{i} \geq \sum_{i=1}^{\infty} \frac{1}{2\sqrt{i}} = \infty,$$

since $\dfrac{2^{2i}}{2\sqrt{i}} \leq \dbinom{2i}{i} \leq \dfrac{2^{2i}}{\sqrt{2i}}$ [MN98, p. 84]. This can also be verified using the Stirling formula, and the resulting sequence diverges. ∎

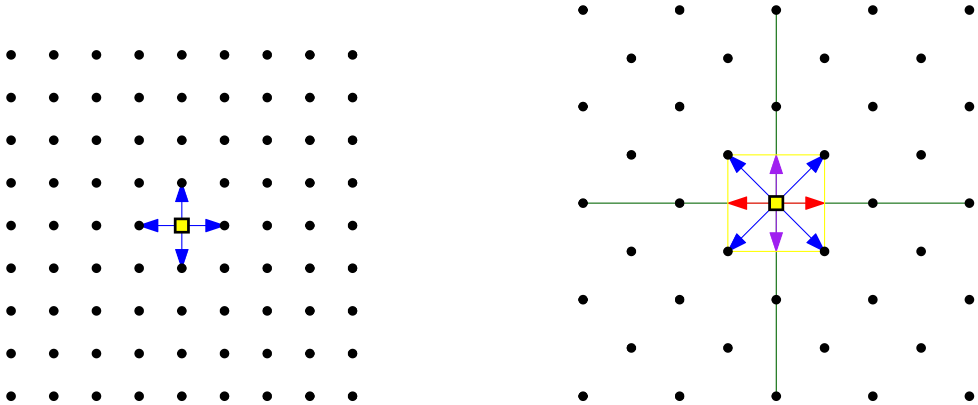

Figure 29.1: A walk in the integer grid, when rotated by 45 degrees, results, in two independent walks on one dimension.

29.1.1.2. Walking on two dimensional grid

A random walk on the integer grid $\mathbb{Z}^d$, starts from a point of this integer grid, and at each step if it is at point $(i_1, i_2, \ldots, i_d)$, it chooses a coordinate and either increases it by one, or decreases it by one, with equal probability.

Lemma 29.1.4. *Consider the infinite random walk on the two dimensional integer grid $\mathbb{Z}^2$, starting from $(0, 0)$. The expected number of times that such a walk visits the origin is unbounded.*

Proof: Rotate the grid by 45 degrees, and consider the two new axes X' and Y', see Figure 29.1.. Let x_i be the projection of the location of the ith step of the random walk on the X'-axis, and define y_i in a similar fashion. Clearly, x_i are of the form $j/\sqrt{2}$, where j is an integer. By scaling by a factor of $\sqrt{2}$, consider the resulting random walks $x'_i = \sqrt{2}x_i$ and $y'_i = \sqrt{2}y_i$. Clearly, x_i and y_i are random walks on the integer grid, and furthermore, they are *independent*. As such, the probability that we visit the origin at the $2i$th step is $\mathbb{P}\Big[x'_{2i} = 0 \cap y'_{2i} = 0\Big] = \mathbb{P}\Big[x'_{2i} = 0\Big]^2 = \Big(\frac{1}{2^{2i}}\binom{2i}{i}\Big)^2 \geq 1/4i$. We conclude, that the infinite random walk on the grid $\mathbb{Z}^2$ visits the origin in expectation

$$\sum_{i=0}^{\infty} \mathbb{P}[x'_i = 0 \cap y'_i = 0] \geq \sum_{i=0}^{\infty} \frac{1}{4i} = \infty,$$

as this sequence diverges. ■

29.1.1.3. Walking on three dimensional grid

In the following, let $\binom{i}{a\ b\ c} = \dfrac{i!}{a!\, b!\, c!}$ denote the multinomial coefficient. The multinomial theorem states that

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{k_1 + k_2 + \cdots + k_m = n \ k_1, k_2, \cdots, k_m \geq 0} \binom{n}{k_1, k_2, \ldots, k_m} \prod_{t=1}^{m} x_t^{k_t}.$$

In particular, we have

$$1^n = (1/3 + 1/3 + 1/3)^n = \sum_{a+b+c=n,\, a,b,c \geq 0} \binom{n}{a\ b\ c} \frac{1}{3^n}. \tag{29.1}$$

Lemma 29.1.5. *Consider the infinite random walk on the three dimensional integer grid $\mathbb{Z}^3$, starting from $(0,0,0)$. The expected number of times that such a walk visits the origin is bounded.*

Proof: The probability of a neighbor of a point (x, y, z) to be the next point in the walk is $1/6$. Assume that we performed a walk for $2i$ steps, and decided to perform $2a$ steps parallel to the x-axis, $2b$ steps parallel to the y-axis, and $2c$ steps parallel to the z-axis, where $a + b + c = i$. Furthermore, the walk on each dimension is balanced, that is we perform a steps to the left on the x-axis, and a steps to the right on the x-axis. Clearly, this corresponds to the only walks in $2i$ steps that arrives to the origin.

Next, the number of different ways we can perform such a walk is $\frac{(2i)!}{a!a!b!b!c!c!}$, and the probability to perform such a walk, summing over all possible values of a, b and c, is

$$\alpha_i = \sum_{\substack{a+b+c=i\\ a,b,c\geq 0}} \frac{(2i)!}{a!a!b!b!c!c!}\frac{1}{6^{2i}} = \binom{2i}{i}\frac{1}{2^{2i}} \sum_{\substack{a+b+c=i\\ a,b,c\geq 0}} \left(\frac{i!}{a!\,b!\,c!}\right)^2 \left(\frac{1}{3}\right)^{2i} = \binom{2i}{i}\frac{1}{2^{2i}} \sum_{\substack{a+b+c=i\\ a,b,c\geq 0}} \left(\binom{i}{a\ \ b\ \ c}\left(\frac{1}{3}\right)^i\right)^2$$

Consider the case where $i = 3m$. We have that $\binom{i}{a\ \ b\ \ c} \leq \binom{i}{m\ \ m\ \ m}$. As such, we have

$$\alpha_i \leq \binom{2i}{i}\frac{1}{2^{2i}}\left(\frac{1}{3}\right)^i \binom{i}{m\ \ m\ \ m} \underbrace{\sum_{\substack{a+b+c=i\\ a,b,c\geq 0}} \binom{i}{a\ \ b\ \ c}\left(\frac{1}{3}\right)^i}_{=1 \text{ by Eq. (29.1)}}.$$

By the Stirling formula, we have

$$\binom{i}{m\ \ m\ \ m} \approx \frac{\sqrt{2\pi i}(i/e)^i}{\left(\sqrt{2\pi i/3}\left(\frac{i}{3e}\right)^{i/3}\right)^3} = c\frac{3^i}{i},$$

for some constant c. As such, $\alpha_i = O\left(\frac{1}{\sqrt{i}}\left(\frac{1}{3}\right)^i \frac{3^i}{i}\right) = O\left(\frac{1}{i^{3/2}}\right)$. Thus,

$$\sum_{m=1}^{\infty} \alpha_{6m} = \sum_i O\left(\frac{1}{i^{3/2}}\right) = O(1).$$

Finally, observe that $\alpha_{6m} \geq (1/6)^2\alpha_{6m-2}$ and $\alpha_{6m} \geq (1/6)^4\alpha_{6m-4}$. Thus,

$$\sum_{m=1}^{\infty} \alpha_m = O(1).$$ ■

29.2. Bibliographical notes

The presentation here follows [Nor98].

References

[MN98] J. Matoušek and J. Nešetřil. *Invitation to Discrete Mathematics*. Oxford Univ Press, 1998.

[Nor98] J. R. Norris. *Markov Chains*. Statistical and Probabilistic Mathematics. Cambridge Press, 1998.