Lecture 5

Continue Chernoff-Hoeffding Bounds

An Application to Routing for Congestion Minimization

A classical problem from both theory and practice.

Given $G=(V,E)$ a directed graph

Let $(s_1,t_1), (s_2,t_2), \ldots, (s_k,t_k)$ be

k source-sink pairs.

The Edge-Disjoint-Paths (EDP) problem asks the following: can we find paths $P_1, P_2, \ldots, P_k$ such that

(i) P_i is an $s_i - t_i$ path

(ii) $P_1, P_2, \ldots, P_k$ are <u>edge</u>-dis<u>joint</u>

A fundamental but difficult problem. We will consider a <u>relaxation</u>. Given G and the pairs find the paths $P_1, \ldots, P_k$ such that no edge is used in too many paths.

We write an LP relaxation

Let $\mathcal{P}_i$ be the set of all $s_i \to t_i$ paths — an exponential set.

$$\min \quad \lambda$$

$$\sum_{p \in \mathcal{P}_i} x_p = 1 \quad \forall i \in [k]$$

$$\sum_{i=1}^{k} \sum_{\substack{p \in \mathcal{P}_i \\ p \ni e}} x_p \leq \lambda \quad \forall e \in E$$

$$x_p \geq 0 \quad p \in \bigcup_i \mathcal{P}_i .$$

We will not discuss how to solve the above LP but it can be done via the Ellipsoid method or by writing a different edge-flow based formulation that I write below where $x(e,i)$ is the flow on edge e for pair i.

$$\min \lambda$$

$$\sum_{e \in \delta^+(s_i)} x(e,i) - \sum_{e \in \delta^-(s_i)} x(e,i) = 1 \quad \forall i \in [k]$$

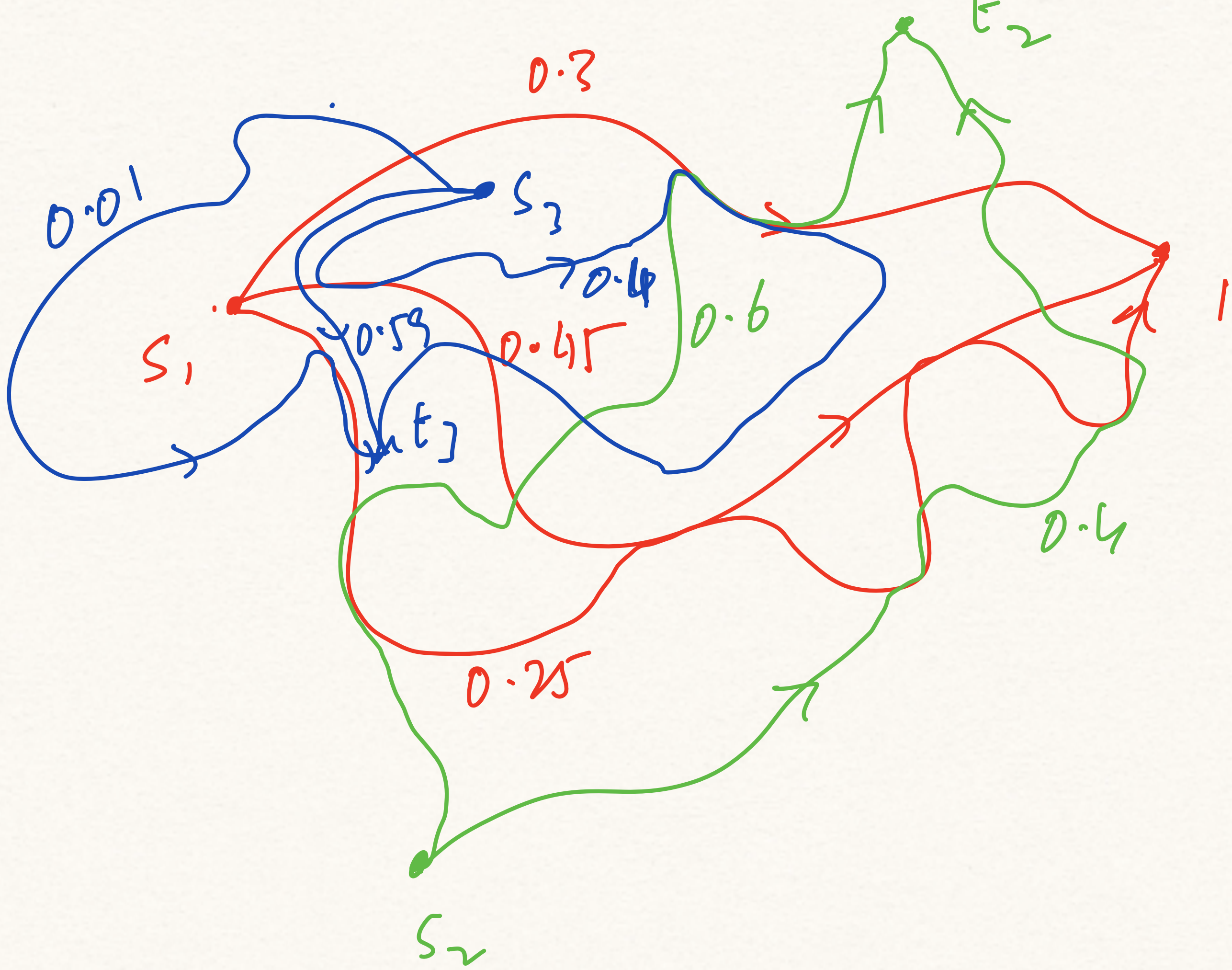
$$\sum_{e \in \delta^+(v)} x(e) - \sum_{e \in \delta^+(v)} x(e) = 0 \quad \forall v \neq \{s_i, t_i\}, \; i \in [k].$$

$$\sum_{i=1}^{k} x(e,i) \leq \lambda \quad \forall e \in E$$

$$x(e,i) \geq 0 \quad e \in E, \; i \in [k].$$

Let λ^* be the optimum value of the LP relaxation.
Note that λ^* can be smaller than 1
So the true lower bound is $\max\{1, \lambda^*\}$.

Can we convert the fractional solution to an integral solution with small "congestion".



Rounding:

1. Solve LP relaxation
2. For each pair (s_i, t_i) independently pick a path $p_i \in P_i$ where p is chosen with probability x_{p_i}.

Raghavan-Thompson were the first to do this rounding and analyze via Chernoff bounds in 1987

<u>Theorem</u>: The algorithm outputs a set of random paths $P_1, \dots, P_k$ such that $\forall i \in [k]$ P_i is an $s_i \to t_i$ path. And with probability $\geq 1 - \frac{1}{\text{poly}(m)}$ the load on any edges is at most

$$O\left(\frac{\log m}{\log\log m}\right) \max\{\lambda^*, 1\}.$$

Proof: Consider any edge e.

Let Y_i be the indicator for P_i, the path chosen for (s_i, t_i) using e.

Let $Y = \sum_{i=1}^{k} Y_i$.

$Pr[Y_i = 1] = x(e,i)$ the total flow on e for commodity i

$= \sum_{\substack{p \in \mathcal{P}_i \\ p \ni e}} x_p$. Why?

$Y_1, Y_2, \ldots, Y_k$ are independent.

$E[Y] = \sum E[Y_i] = \sum_{i=1}^{k} x(i,e) \leq 1$

Hence by Chernoff bound

$$\Pr\left[Y \geq c \frac{\log(m)}{\log\log(m)} \right] \leq \frac{1}{m^{c'}}$$

~~to~~ for sufficiently large constants c and c'.

Then we apply union bound over all the edges and since we have m edges tha

$$\Pr\left[\text{load on any edge } e \geq c \frac{\ln m}{\ln \ln m} \right] \leq m \cdot \frac{1}{m^{c'}} \leq \frac{1}{m^{c'-1}}. \quad \square$$

Exercise: You can use Chernoff bounds to prove two related bounds

(i) $\exists\, c, c'$ such that for $\varepsilon \in (0,1)$.

$$\Pr\left[Y \geq (1+\varepsilon)\lambda^* + \frac{c \log m}{\varepsilon^2} \right] \leq \frac{1}{m^{c'}}$$

Thus when $\lambda^* = \Omega(\log m)$ we get a very good approximation.

(ii) When $\lambda^* \geq c \log m$ ~~the~~

$$\Pr\left[Y \geq \lambda^* + \sqrt{c \log k \, \lambda^*} \right] \leq \frac{1}{k^{c'}}.$$

Note the preceding bound has no multiplicative factor on λ^*.

Additive Chernoff Bound

To motivate this bound consider the random walk on the line

we had $Y = \sum_{i=1}^{n} X_i$

where $X_i \in \{-1, 1\}$ and $E[X_i] = 0$

and hence $E[Y] = 0$.

In this setting we cannot expect a multiplicative Chernoff bound.

We will state a general bound that handles this kind of setting

Hoeffding Bound:

Let $X = \sum_{i=1}^{n} X_i$ where

(i) X_i are independent

(ii) $X_i \in [a_i, b_i]$ $\forall i \in [n]$

(iii) $E[X_i] = 0$ $\forall i \in [n]$

Then, for $\eta > 0$

$$\Pr[X \geq \eta] \leq e^{-\frac{\eta^2}{2\sum_i (b_i - a_i)^2}}$$

and for $\eta < 0$

$$\Pr[X \leq \eta] \leq e^{-\frac{\eta^2}{2\sum_i (b_i - a_i)^2}}.$$

Comments: Suppose $X_i \in [-1, 1]$

(i) Then $\sum_{i=1}^{n} (b_i - a_i)^2 = 4n$.

(ii) Why assume $E[X_i] = 0$. We can replace X_i by. $Y_i = X_i - E[X_i]$

and $E[Y_i] = 0$ and if $X_i \in [a_i, b_i]$

then $Y_i \in [a_i - E[X_i], b_i - E[X_i]]$

and hence the term $(b_i' - a_i')$ does not change.

$\Rightarrow$ without assuming $E[X_i] = 0$ we have

$$\Pr\left[X - E[X] \geq t\right] \leq e^{-\frac{t^2}{2\sum_{i=1}^{n}(b_i - a_i)^2}}$$

which is the standard form.

Proof: As before we consider e^{tX} for a parameter $t > 0$

$$\Pr[X \geq \eta] = \Pr\left[e^{tX} \geq e^{t\eta}\right]$$

$$\leq \frac{E[e^{tX}]}{e^{t\eta}} \quad \text{by Markov.}$$

So it boils down to estimating/upper bounding $E[e^{tX}] = \prod_{i=1}^{n} E[e^{tX_i}]$

and choosing the best t..

How do we bound e^{tX_i}

We will only sketch the argument.

e^{ty} is a convex function in the interval $[a_i, b_i]$ (over the entire real line).

We know $E[X_i]=0$ and

$X_i \in [a_i, b_i]$

What is the distribution that maximizes e^{tX_i} since that is what gives us the weakest bound?

Due to convexity it turns out that we should put all the probability mass on the extremes of the interval $[a_i, b_i]$. So the worst case is when $X_i \in \{a_i, b_i\}$ subject to $E[X_i] = 0$ note $a_i \leq 0$ and $b_i \geq 0$.

Say X_i is b_i with $p = \frac{-a}{b-a}$

and a_i with prob $1-p = \frac{b}{b-a}$.

One can prove above by convexity.

Assuming above

$$E\left[e^{tX_i}\right] \leq p e^{tb_i} + (1-p) e^{ta_i}$$

* (not completely obvious).

By calculus we can show that

$$\leq e^{\frac{t^2 (b_i - a_i)^2}{8}}.$$

Assuming above we have

$$E\left[e^{tX}\right] \leq e^{\frac{t^2}{8}\sum_{i=1}^{n}(b_i-a_i)^2}$$

Thus $\Pr\left[X \geq \eta\right] \leq \dfrac{E\left[e^{tX}\right]}{e^{t\eta}}$

$$\leq e^{\frac{t^2 \sum (b_i-a_i)^2}{8} - t\eta}.$$

minimizing over t we have

$$\frac{t}{4}\sum_{i=1}^{n}(b_i-a_i)^2 = \eta$$

$$t^* = \frac{4\eta}{\sum_{i=1}^{n}(b_i-a_i)^2}.$$

plugging in

$$\frac{t^{*2}}{8} \sum_i (b_i - a_i)^2 - t^* \eta$$

$$= \frac{16 \eta^2}{8 \sum_i (b_i - a_i)^2} - \frac{4 \eta^2}{\sum_i (b_i - a_i)^2}$$

$$= \frac{-2 \eta^2}{\sum_{i=1}^{n} (b_i - a_i)^2}$$

Thus

$$\Pr [X > \eta] \leq e^{-\frac{2\eta^2}{\sum_{i=1}^{n} (b_i - a_i)^2}}.$$

Lower Tail is similar.

Application

Random Walk on the Line.

$$X = \sum_{i=1}^{n} X_i \qquad X_i \in \{-1, 1\}$$

$$E[X_i] = 0$$

$$b_i = 1 \quad a_i = -1$$

$$\Pr[X > t\sqrt{n}] \leq e^{-\frac{t^2 n}{8n}} \leq e^{-\frac{t^2}{8}}.$$

Similarly $\Pr[X < t\sqrt{n}] \leq e^{-\frac{t^2}{8}}.$