Homework 4

CS 574: Randomized Algorithms, Fall 2025
Due: Thursday, Nov 13th 2025 at noon

Instructions and Policy:

- Each homework can be done in a group of size at most two. Only one homework needs to be submitted per group. However, we recommend that each of you think about the problems on your own first.
- Homework needs to be submitted in pdf format on Gradescope. See `https://courses.grainger.illinois.edu/cs374al1/sp2025/hw-policies.html` for more detailed instructions on Gradescope submissions.
- Follow academic integrity policies as laid out in student code. You can consult sources but cite all of them including discussions with other classmates. Write in your own words. See the site mentioned in the preceding item for more detailed policies.

Optional Problem. In a connected graph G , an edge is called a bridge if the removal of the edge disconnects the graph. Let G be a connected graph with n vertices and m edges. Let (u, v) be any edge in G . For the simple random walk on G, show that $h_{u,v} + h_{v,u} = 2m$ if and only if the edge (u, v) is a bridge. Recall that $h_{u,v}$ is the expected time for the walk to visit v starting at u.

Problem 1. Exercise 7.10 in Mitzenmacher-Upfal book on random walks.

Problem 2. Suppose G is a n-vertex non-bipartite regular graph with expansion α where $\alpha = \Omega(1)$. Prove that the cover time is $O(n \log n)$.

Problem 3. Earlier in the course we saw the congestion minimization problem where the goal was to connect a given set of k source-vertex pairs $(s_1, t_1), \ldots, (s_k, t_k)$ in a directed graph $G = (V, E)$ by paths to minimize the congestion of the paths. We used randomized rounding and Chernoff bounds to obtain an $O(\log n/ \log \log n)$ upper bound on the congestion with respect to the fractional routing. Here we consider a generalization. For each pair (s_i, t_i) we are now given an explicit list $\mathcal{P}_i$ of s_i-t_i paths and an integer ℓ_i where $\ell_i \leq |\mathcal{P}_i|$. We need to choose *exactly* ℓ_i paths for each (s_i, t_i) from $\mathcal{P}_i$ so that we minimize $\max_{e \in E} \alpha(e)$ where $\alpha(e)$ is the total number of paths that use e from the chosen set of paths (recall that we are choosing paths for each pair and paths from same pair or paths from different pairs may use e).

- Write a simple LP relaxation to choose the paths for each pair.
- Adapt pipage rounding that we saw in lecture for Max k-Cover to obtain a randomized algorithm to choose the paths.
- Use the negative correlation property of pipage rounding that to obtain an $O(\log n/ \log \log n)$-approximation for minimum congestion routing in this more general setting.

Problem 4. Exercise 13.20 in Mitzenmacher-Upfal book on martingale analysis for balls and bins.

Problem 5. Extra Credit: Consider an urn U with n balls, where αn of them are red, and $(1-\alpha)n$ are blue. Assume α is a small constant (say 0.01). Consider the following process: In the i th step, you randomly pick a set S_i of three balls from the urn. Let C_i be the majority of the colors of balls in S_i. You next put the balls of S_i back into U, randomly throw away a ball from U, and add to U a new ball of color C_i. The game is repeated till all the balls are of the same color. Give an upper and lower bounds on the number of rounds you have to play this game till all balls have the same color. (The bounds should be reasonably tight.)