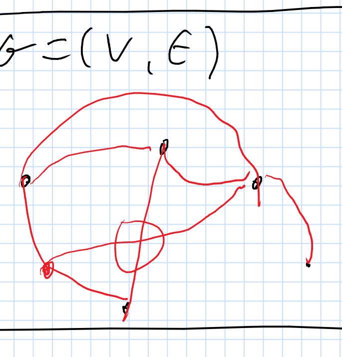
4/8/2021

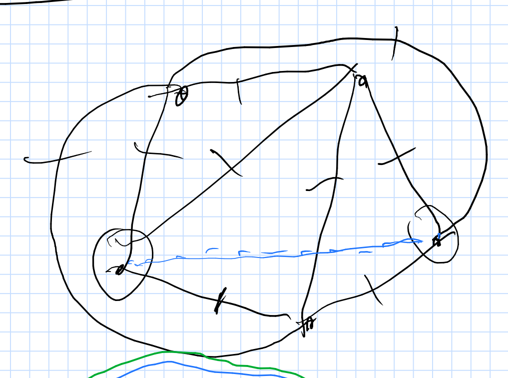
planar graphs

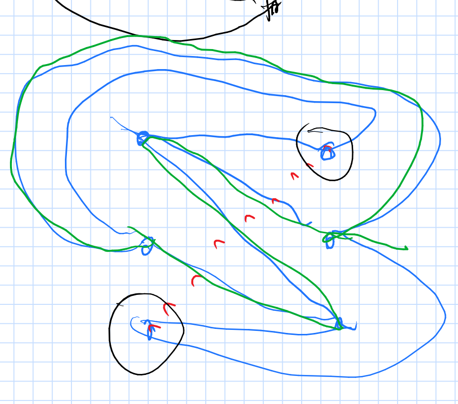
$G=(V,E)$

planar graph – a graph that can be drawn in the plane.

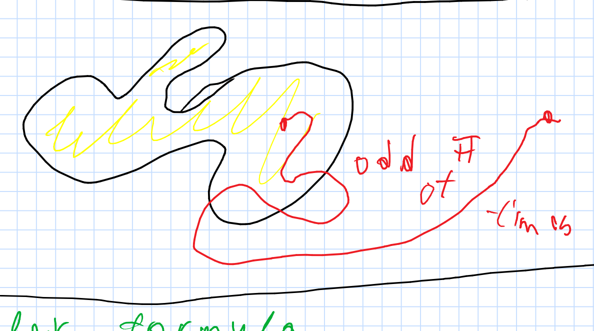
$n=|V|$ $G=(V,E)$



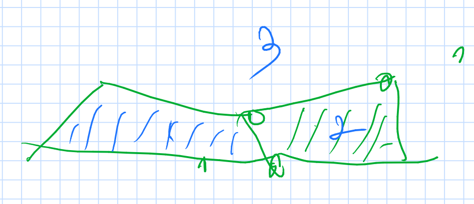


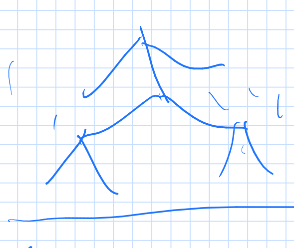


Jordan curve theorem



Euler formula





G

n: # vertices

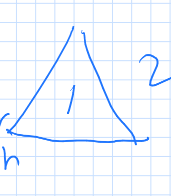
m: edges

f = faces

Euler formula

$n-m+f=2$

for any connected planar graph



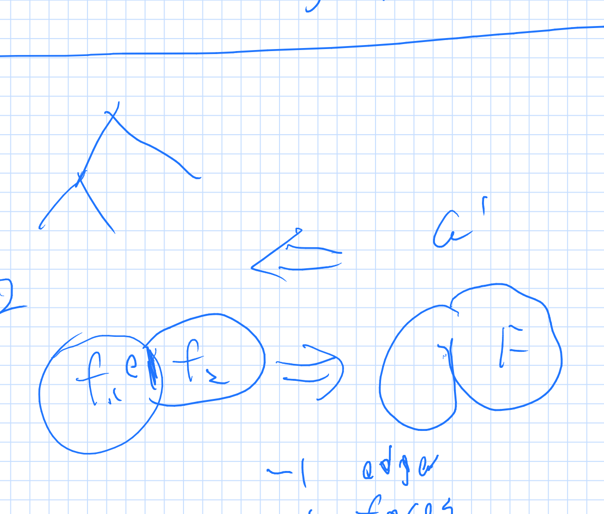
proof

$f=1$

$m=n-1$

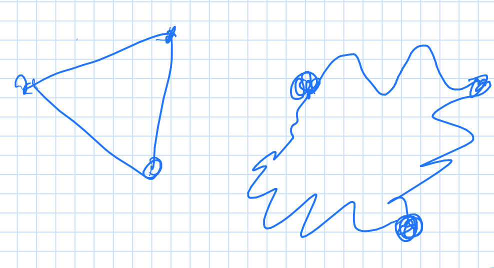
$n-m+f=2$

$f>1$



$n-m+f=2$

A triangulation is a planar graph where all faces are triangles.



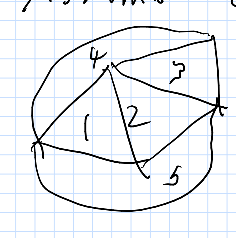
claim

every planar graph can be triangulated.

claim

A planar graph with n vertices has at most $3n-6$ edges.

proof Assume G is a triangulation



n vertices

f faces

m edges

$3f=2m \Rightarrow f=\frac{2}{3}m$

$n-m+f=2$

$m=n+f-2=n+\frac{2}{3}m-2$

$\frac{1}{3}m=n-2 \Rightarrow m=3n-6$ ⟸ triangulation

$m\le 3n-6$ ∎

K_n $|E(K_n)|=\frac{n(n-1)}{2}=\binom{n}{2}\approx\frac{n^2}{2}$

sparse $m\le 3n-6$

claim

Planar graphs can be colored by 6 colors.

Lemma In any planar graph there is a vertex of degree at most 5.

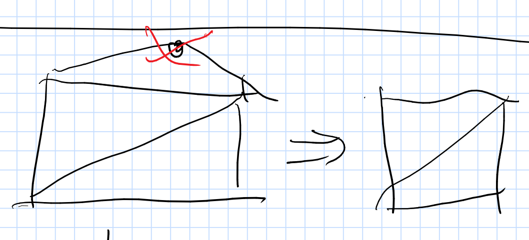
Proof

if false $|E(G)|=\frac{1}{2}\sum_{v\in V}d(v)\le 3n-6$

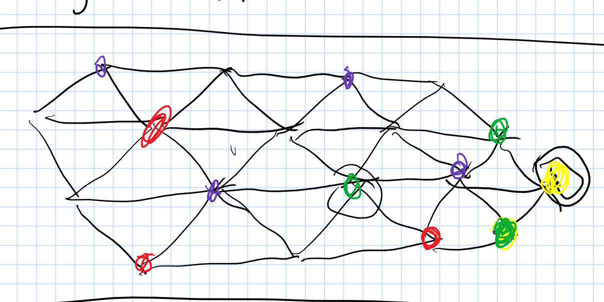
$|V|=n$

$\sum_{v\in V}d(v)\le 6n-12$

$\Rightarrow \exists v$ $d(v)\le 5$. ∎



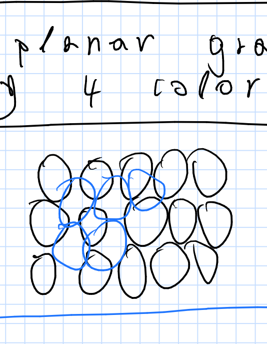
5-degenerate



Planar graphs can be colored by 5 colors.

Four color theorem

Any planar graph can be drawn using 4 colors.



G 3-colorable

$\approx\sqrt{n}$ $n^{1/4}$

claim K_5 is not a planar graph.

proof $n=5$

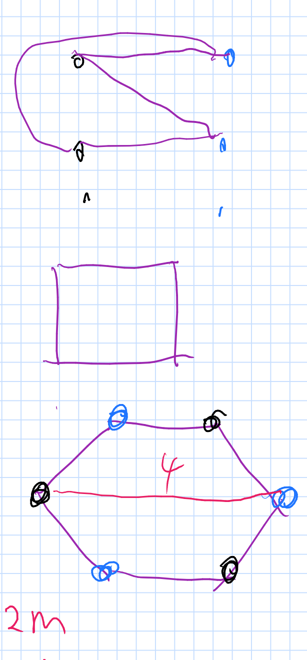
$m=\frac{5\cdot 4}{2}=10$

Euler formula $10=m\le 3n-6=15-6=9$.

A contradiction.

$K_{3,3}$

Euler formula for bipartite graphs



$3f=2m$

$4f=2m$

claim for any bipartite planar graph, the number of edges is $\le 2n-4$

proof

$4f=2m \Rightarrow f=\frac{m}{2}$

$n-m+f=2$

$m=n+f-2 \Rightarrow \frac{m}{2}=n-2 \Rightarrow m\le 2n-4$

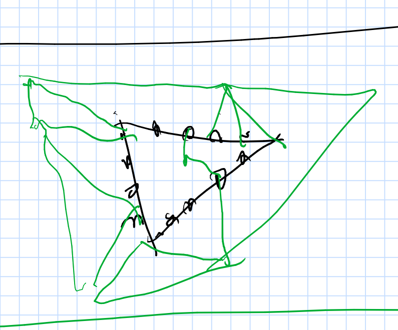
$K_{3,3}$ is not planar.

$n=6$

$m=9\le 2n-4=12-4=8$. impossible

Kuratowski's theorem

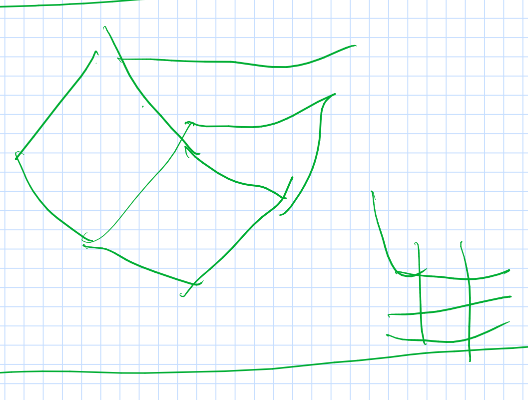
G is planar ⟺ G does not contain a subdivision of $K_{3,3}$ or K_5

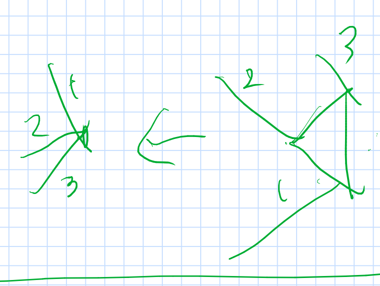


One can decide if a graph is planar in $O(n)$

Wagner theorem

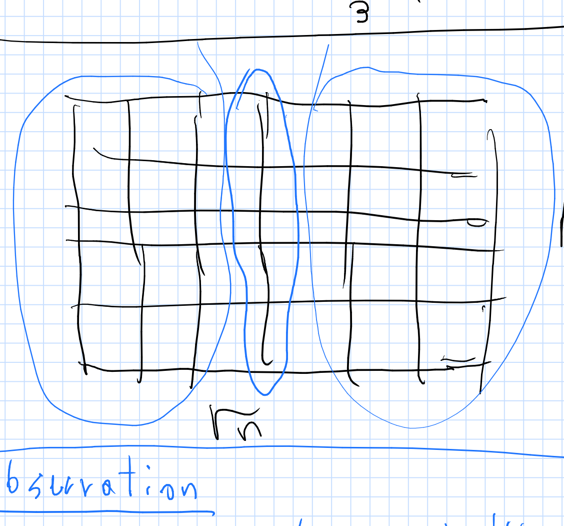
G is a planar graph ⟺ does not have $K_{3,3}$ or K_5 as a minor.





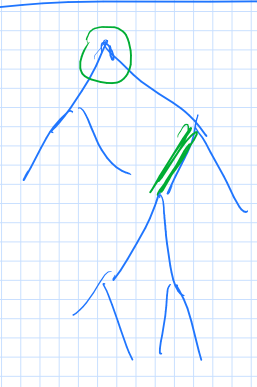
Planar separators

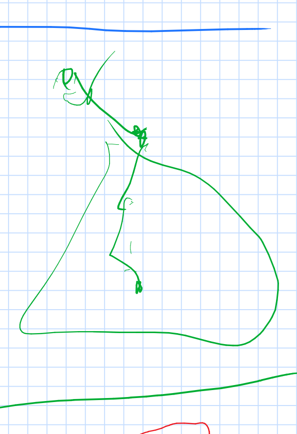
Theorem For any planar graph with n vertices there is a set S of $O(\sqrt{n})$ vertices s.t. $G\setminus S$ is disconnected and every connected component has at most $\frac{2}{3}n$ vertices.



observation

For any tree with n vertices and degree at most 3, there is an edge that its removal breaks the tree into parts, s.t. each part has at most $\frac{2}{3}n$ vertices.





G planar triangulated