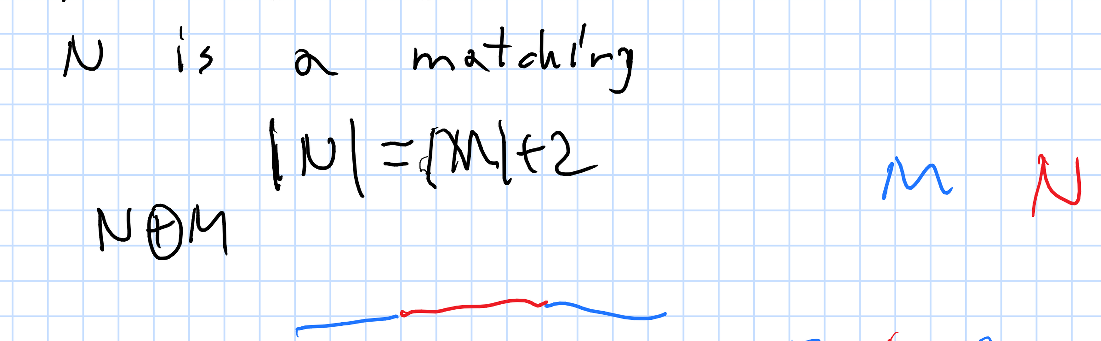
3/30/21

more on matchings

Lemma
M matching
π: shortest augmenting path for M
π': shortest aug path for $M \oplus \pi$
then
$$|\pi'| \geq |\pi| + 2|\pi \cap \pi'|$$

Proof
$N = M \oplus \pi \oplus \pi'$
N is a matching
$|N| = |M| + 2$
$N \oplus M$

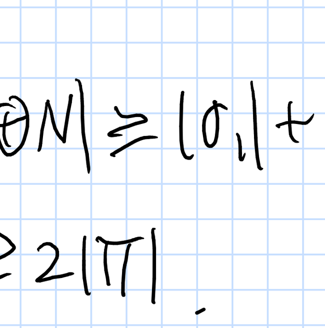
M N



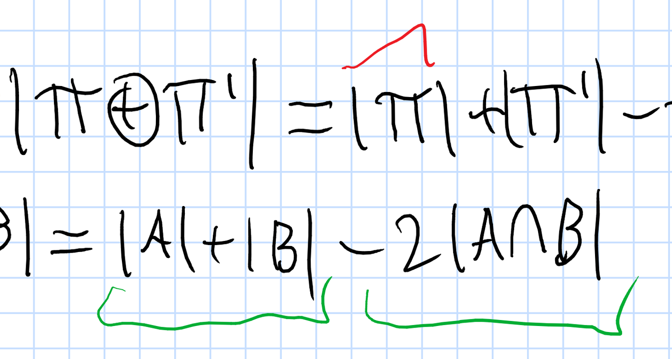
$|\sigma_1| \geq |\pi|$
$|\sigma_2| \geq |\pi|$

$$|\pi \oplus \pi'| = |M \oplus M \oplus \pi \oplus \pi'|$$
$$= |M \oplus \underbrace{(M \oplus \pi \oplus \pi')}_{N}| = |M \oplus N| \geq |\sigma_1| + |\sigma_2| \geq 2|\pi|.$$

$$2|\pi| \leq |\pi \oplus \pi'| = |\pi| + |\pi'| - 2|\pi \cap \pi'|$$
$$|A \oplus B| = |A| + |B| - 2|A \cap B|$$

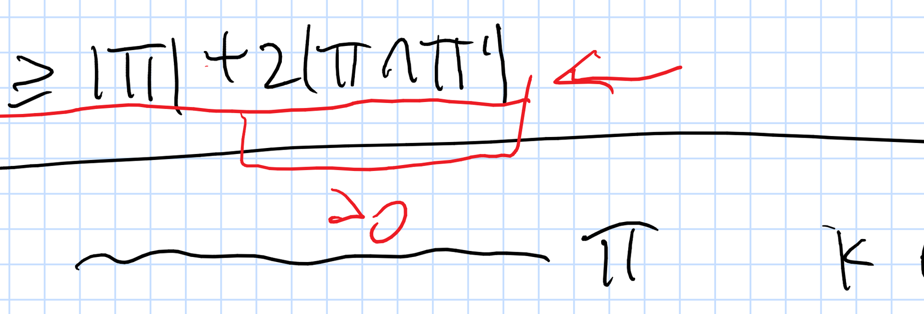


$$|\pi'| \geq |\pi| + 2\underbrace{|\pi \cap \pi'|}_{\geq 0}$$

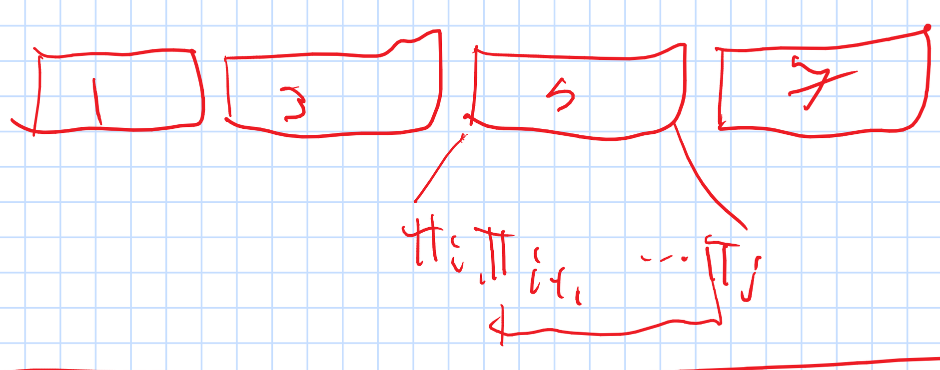


k edges
$|\pi| = |\pi'|$

Claim π and π' are disjoint.



$\pi_1, \pi_2, \pi_3 \ldots \pi_t$



Claim The shortest aug path algorithm, augments along a set of aug paths, such that all the paths of the same length are disjoint.

Edmunds-Karp algorithm

G, $M \leftarrow \{\}$
For $k = 1, 3, 5, 7, \ldots \sqrt{n}$ do
  $P_k \leftarrow$ maximal set of aug paths that are disjoint — $O(m)$
  For $\pi \in P_k$ do
    $M \leftarrow M \oplus \pi$
while $\exists$ aug path π do
  $M \leftarrow M \oplus \pi$

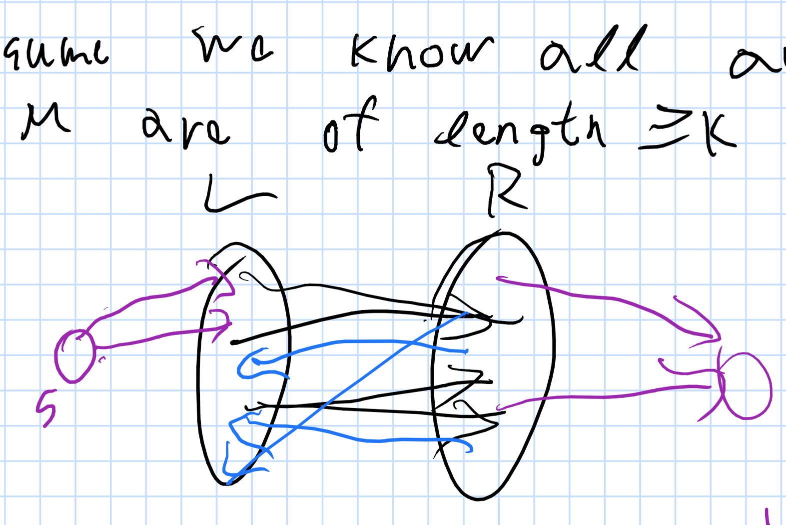
$O(m)$ times
$O(\sqrt{n})$ times

$O(m\sqrt{n})$

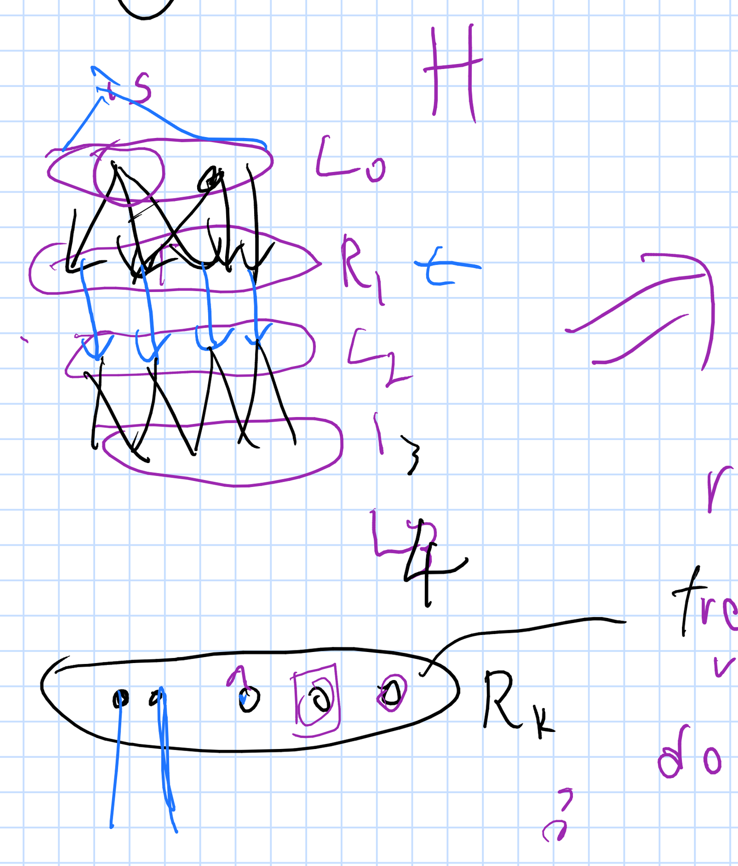
G, M — Bipartite graph

Assume we know all aug paths for M are of length $\geq k$

BFS computes layers



H H^{rev}



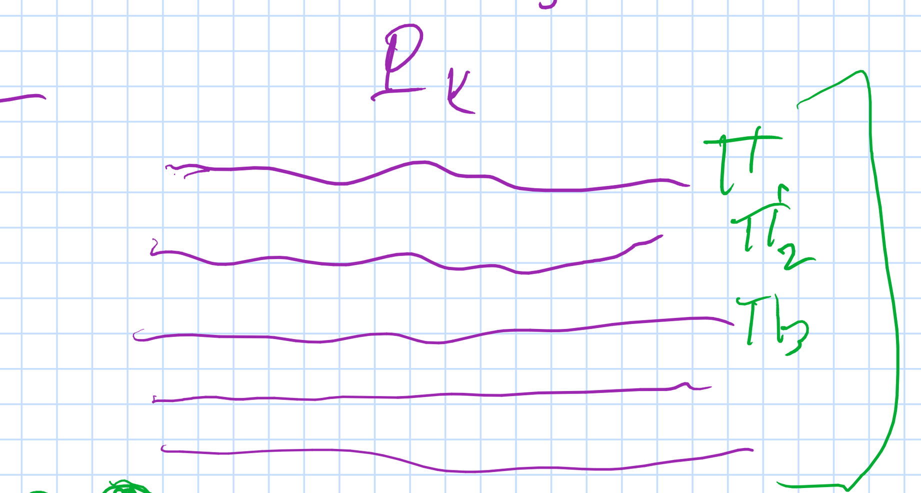
reverse graph
free vertex
do DFS in the reverse graph

$O(m)$

$\Rightarrow$ P_k Get collection of aug paths of length k that are disjoint.

Claim P_k is a maximal set of disjoint aug path of length k.

Proof



$\Rightarrow$ Edmunds-Karp indeed output maximum matching

Claim
Let M be a matching s.t. all aug paths are of length $\geq \sqrt{n}$.
$\Rightarrow |M| \geq |\underbrace{\text{Maximum Matching}(G)}_{N}| - \sqrt{n}$

Proof
$M \oplus N$ = collection of paths that contains at least k aug paths for M. disjoint
$k = |N| - |M|$
$k(\sqrt{n} + 1) \leq n$
$k \leq \sqrt{n}$

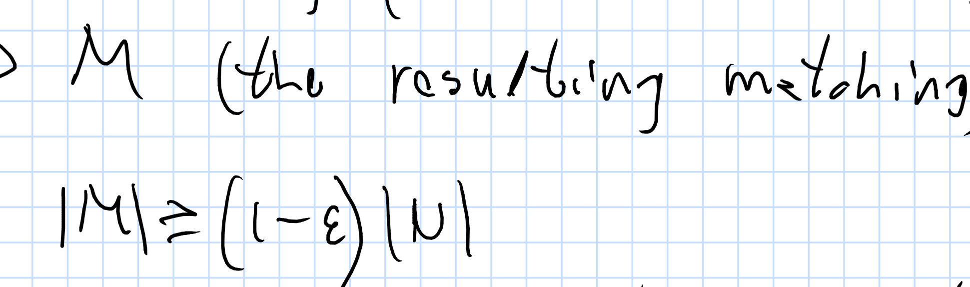
Run EK algorithm for $\lceil 1/\varepsilon \rceil$ iterations $\varepsilon \in [0,1]$
$\Rightarrow$ all aug paths are of len $\geq 1/\varepsilon$
$\Rightarrow$ M (the resulting matching)

$$|M| \geq (1-\varepsilon)|N|$$

Result One can compute a matching of size $(1-\varepsilon)\text{opt}$ in $O(\frac{m}{\varepsilon})$.

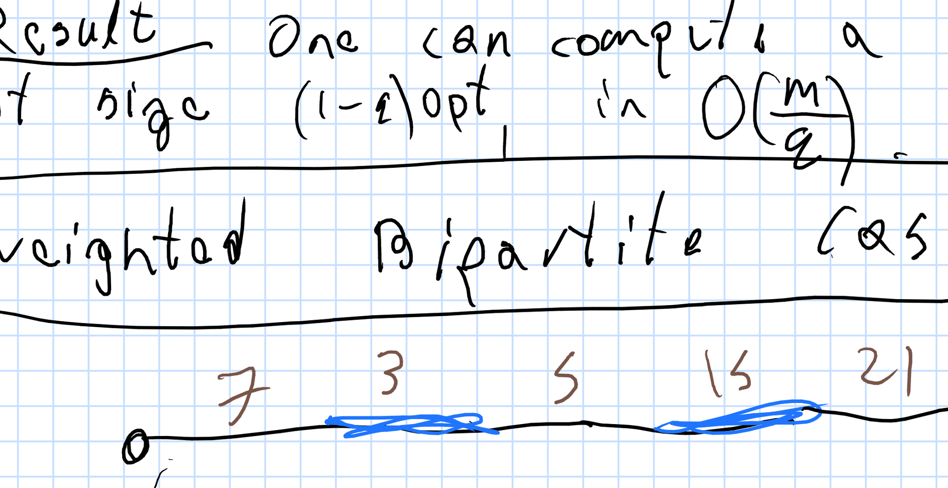
$\boxed{O(m\sqrt{n}) \text{ EK}}$

weighted Bipartite case



aug path

$$7 - 3 + 5 - 15 + 21 > 0$$



matching edges
negative weight
– flip prices of edges
– shortest path

Lemma If you always augment along the heaviest path, then the graph we use to compute the aug path does not contain negative cycles.
$\Rightarrow$ Bellman Ford $O(mn)$

Repeatedly aug along heaviest aug path.

$O(mn^2)$ time

Theorem Computing max weight matching in bipartite graph can be done in $O(mn^2)$ time.