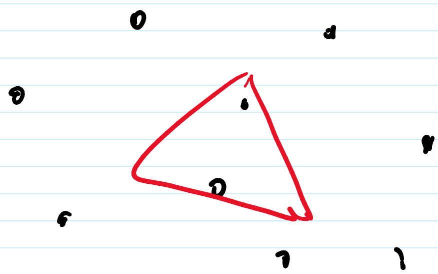
December 5, 2024 3:08 PM

Final Exam Dec 16 Mon 1:30-4:30 LuMEB 3101

See web site. for past finals etc.

2 cheat sheets

Triangle Range Searching



Partition tree

Willard '82 $S(n) = O(n)$

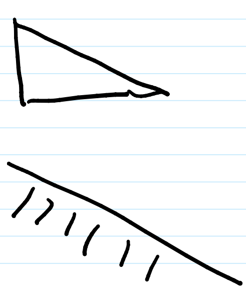
$Q(n) = O(n^{\log_4 3}) = O(n^{0.79})$

⋮

Matoušek '91 $S(n) = O(n)$

$Q(n) = O(n^{\frac{1}{2}+\varepsilon})$

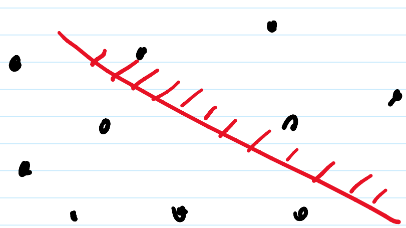
$n^{1-\frac{1}{d}+\varepsilon}$



Next: aim for $S(n) = ??$

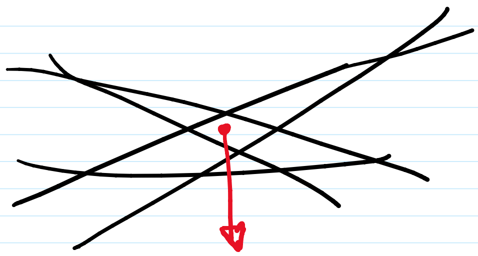
$Q(n) = O(\log n)$

focus on halfplane case (counting)



dual

$\Rightarrow$



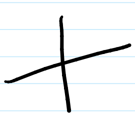
count # lines below query pt q.

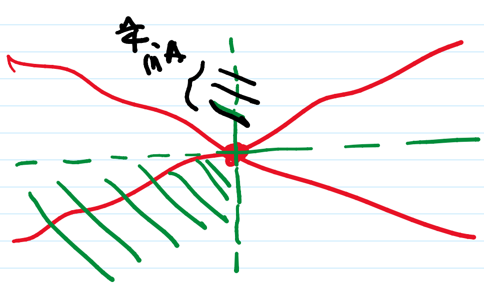
Cutting Tree

Cutting Lemma Given n lines in $\mathbb{R}^2$,
can cut $\mathbb{R}^2$ into 4 cells s.t.
each cell intersects $\leq \cancel{\tfrac{7}{8}}\tfrac{3}{4}n$ lines

Megiddo's LP in 3D

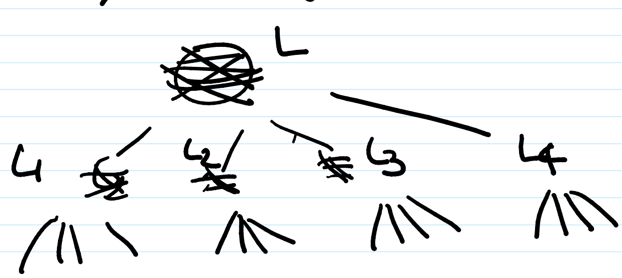
Pf: let m = median slope
A = lines w. slope $\leq m$
B = lines w. slope $> m$

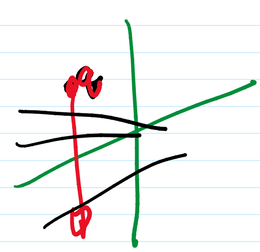




$\square$

recurse $\Rightarrow$ cutting tree





$$Q(n) = 1 \cdot Q\left(\tfrac{3}{4}n\right) + O(1)$$

$$\Rightarrow O(\log_{4/3} n) = \boxed{O(\log n)}$$

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$$\Rightarrow \quad O(\log_{4/3} n) = \boxed{O(\log n)}$$

$$S(n) = 4\, S\left(\tfrac{3}{4}n\right) + O(1)$$

$$\Rightarrow \quad O\left(n^{\log_{4/3} 4}\right) = \boxed{O(n^{4.82})}$$

Clarkson's Cutting Lemma ('87)

Given n lines in $\mathbb{R}^2$, $\leftarrow cr^2 \cancel{\log^2} r$

cut into $\underbrace{\le cr^2}$ cells $\leftarrow cr^d$ in $\mathbb{R}^d$

s.t. each intersects $\le \frac{n}{r}$ lines

r const

$$\Rightarrow \quad S(n) = cr^2\, S\left(\tfrac{n}{r}\right) + O(1)$$

$$\Rightarrow O\left(n^{\frac{\log(cr^2)}{\log r}}\right)$$

$$= O\left(n^{\frac{\log c + 2\log r}{\log r}}\right)$$

$$= O\left(n^{2 + \frac{\log c}{\log r}}\right)$$

$$= \boxed{O(n^{2+\varepsilon})} \leftarrow n^{d+\varepsilon}$$

$$Q(n) = Q\left(\tfrac{n}{r}\right) + O(1)$$

$$\Rightarrow Q(n) = \boxed{O(\log n)}$$

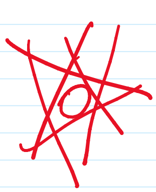
Proof of Clarkson's Lemma:

Very Simple Randomized Algm:

1. take random sample R of L of br lines (prob. $\frac{br}{n}$)

2. return a triangulation $T(R)$ of the arrangement of R

$\Rightarrow O((br)^2)$ cells

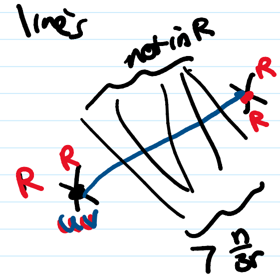


$\Pr[\text{alg'm errs}] \le \Pr[$some cell of $T(R)$ intersects $> \frac{n}{r}$ lines$]$

$$\Pr[\text{alg'm errs}] \le \Pr[\text{some cell of } T(R) \text{ intersects} > \cancel{?} \text{ lines}]$$

$$\le \Pr[\text{some edge of } T(R) \text{ intersects} > \tfrac{n}{3r} \text{ lines}]$$

Fix line seg uv that intersects $> \frac{n}{3r}$ lines



$$\Pr[uv \text{ is an edge of } T(R)]$$

$$\le \left(\frac{br}{n}\right)^4 \left(1 - \frac{br}{n}\right)^{\frac{n}{3r}}$$

$$\le \left(\frac{br}{n}\right)^4 e^{-\frac{br}{n}\cdot\frac{n}{3r}} \qquad (1-x \le e^{-x})$$

$$= \left(\frac{br}{n}\right)^4 \frac{1}{e^{b/3}}$$

$$\Rightarrow \Pr[\text{alg'm errs}] \le n^4 \cdot \left(\frac{br}{n}\right)^4 \frac{1}{e^{b/3}}$$

$$= \frac{b^4 r^4}{e^{b/3}} \qquad b = 15 \ln r$$

$$= O\left(\frac{r^4 \log^4 r}{r^5}\right)$$

$$\to 0.$$

To summarize:

$$\underbrace{\begin{array}{l} S(n) = O(n) \\ Q(n) = O(n^{\frac{1}{2}+\varepsilon}) \end{array}}_{\text{partition tree}} \qquad \underbrace{\begin{array}{l} S(n) = O(n^{2+\varepsilon}) \\ Q(n) = O(\log n) \end{array}}_{\text{cutting tree}}$$

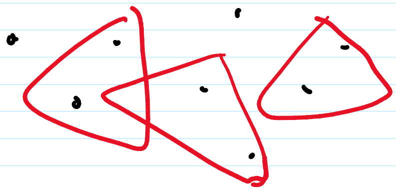
Can combine to get a trade-off

cutting tree

$\left(\frac{n}{s}\right)^{2+\varepsilon}$

$$P(n) = \boxed{\begin{array}{l} S(n) = O\left(\left(\frac{n}{s}\right)^{2+\varepsilon} \cdot s\right) \\ Q(n) = O\left(\log n + s^{\frac{1}{2}+\varepsilon}\right) \end{array}}$$

Ex: given n pts & n triangles,
want to count # pts inside triangle



$$\tilde{O}\left(\left(\frac{n}{s}\right)^{2\cancel{+\varepsilon}} \cdot s + n\, s^{\frac{1}{2}\cancel{+\varepsilon}}\right) \quad \text{ignoring } n^{\varepsilon}$$

$$= \tilde{O}\left(\frac{n^2}{s} + n\sqrt{s}\right)$$

$\frac{n^2}{s} = n\sqrt{s}$

$s^{3/2} = n$

set $s = n^{2/3}$

$$\Rightarrow \boxed{\tilde{O}(n^{4/3})}$$

Similar $n^{4/3}$ algms for many problems

e.g. segment intersection counting
distance selection
EMST in $\mathbb{R}^2$
etc.