$$Q_2(n) = O\Big(\underbrace{(\#\text{ slabs } R \text{ is short})}_{2\log n} * \underbrace{Q_1(n)}_{\log n}\Big)$$

$$= \boxed{O(\log^2 n)} \quad (+k \text{ for reporting})$$

$\downarrow$

$O(\log n)$ by using more ptrs

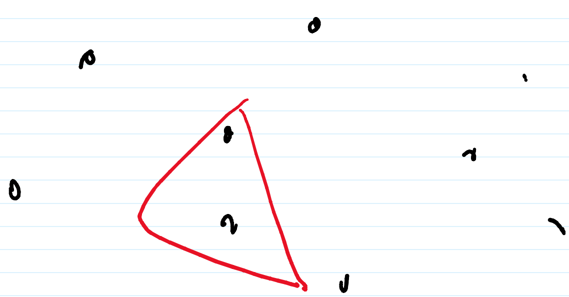
higher d:

$$S_d(n) = \boxed{O(n \log^{d-1} n)}$$

$$Q_d(n) = \boxed{O(\log^{d-1} n)}$$

$$P_d(n) = O(n \log^{d-1} n)$$

General Case of Triangular Range Search



Method 1: Willard's Partition Tree ('82)

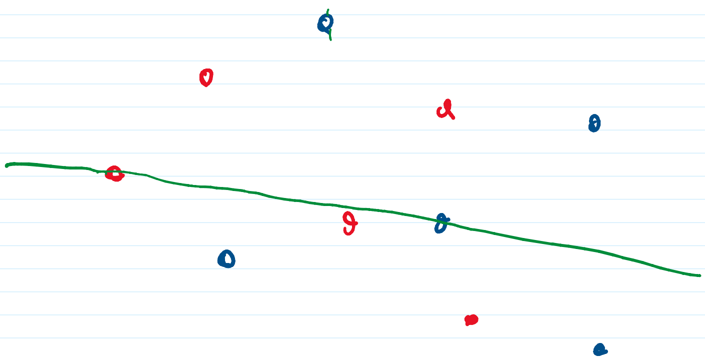
Ham-Sandwich Thm (2D discrete version)

Given two point sets $P_1, P_2 \in \mathbb{R}^2$,
there exists a line that simultaneously bisects P_1, & P_2

there exists a line that simultaneously bisects P_1, & P_2

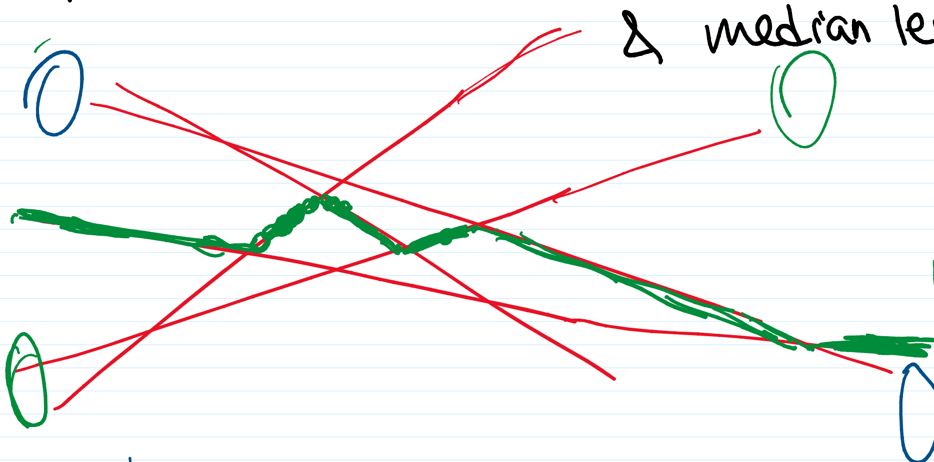
$|P_1|/2$ pts on each side of line

$|P_2|/2$ pts on each side of line

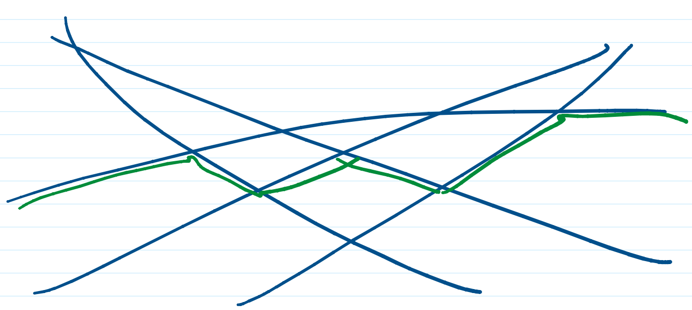


Proof: say $|P_1|, |P_2|$ odd

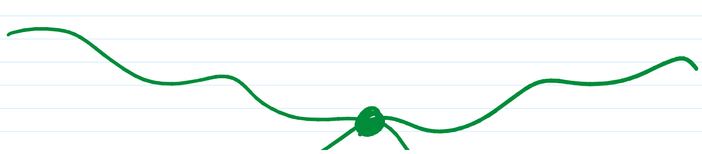
Duality $\Rightarrow$ Given two line arrangements $\mathcal{A}_1, \mathcal{A}_2$, to show there exists a point on median level of $\mathcal{A}_1$ & median level of $\mathcal{A}_2$



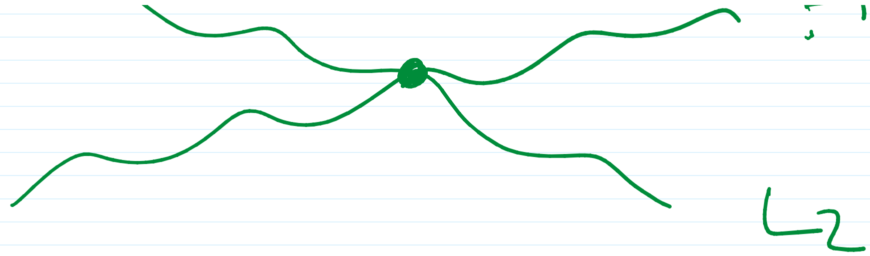
$L_1 =$ median level of $\mathcal{A}_1$

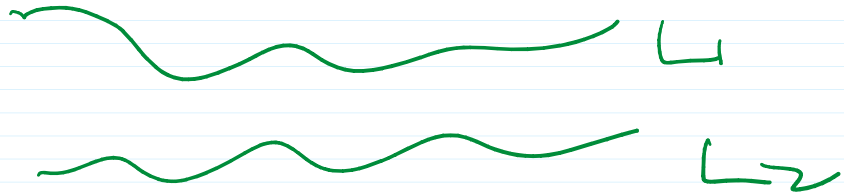


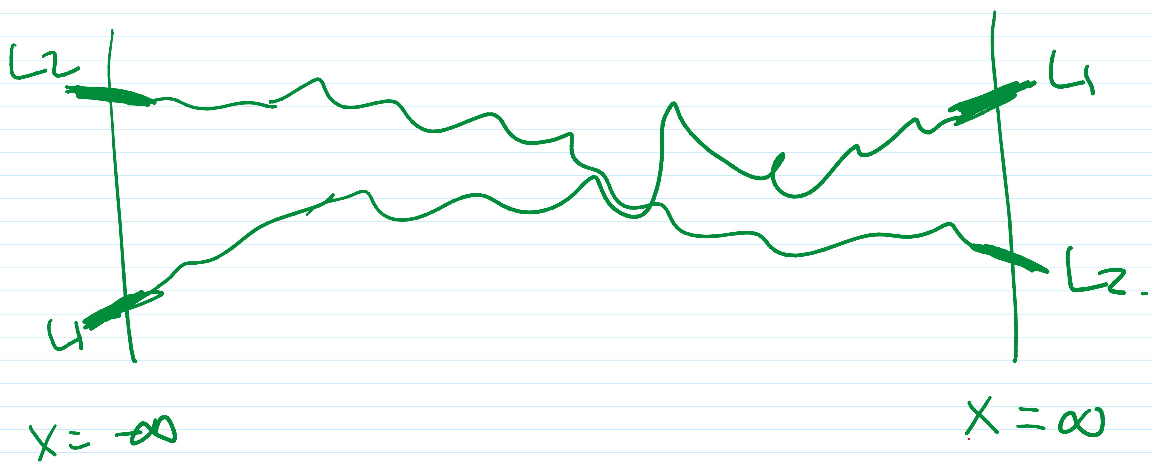
$L_2 =$ median level of $\mathcal{A}_2$



L_1

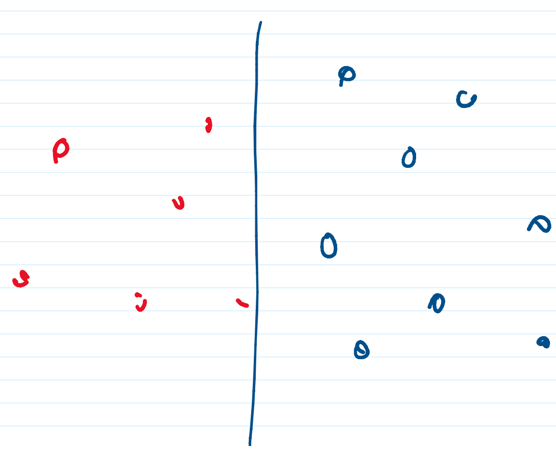


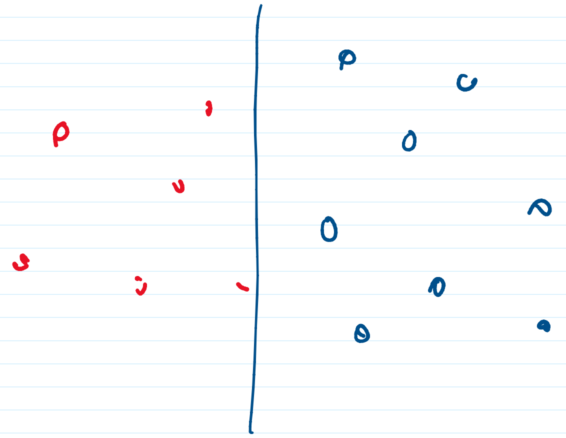




by intermediate value thm,
L_1 & L_2 must intersect.! $\square$

Rmk: can be constructed in $O(n)$ time
by Megiddo (like LP, prune-and-search)
assuming P_1, P_2 are separated by line

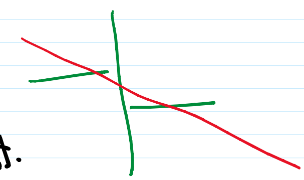




assuming P_1, P_2 are separated by line

OPEN: how many vertices on median level?
$O(n^{4/3})$ known by Dey '97

A Partition Thm Given set P of n pts in $\mathbb{R}^2$,

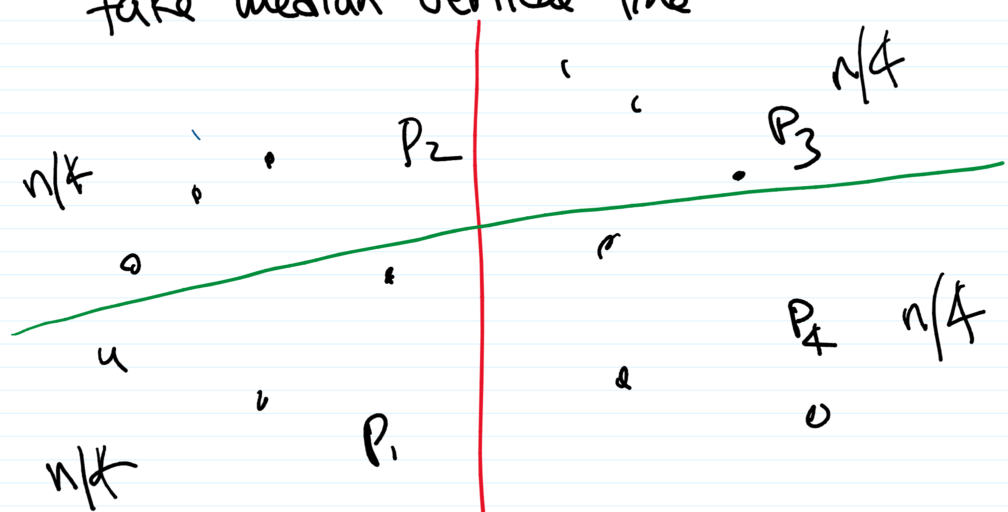


can partition P into 4 subsets of size $\frac{n}{4}$ s.t.

(i) each subset is contained in a region

(ii) any line crosses ≤ 3 regions.

Pf: take median vertical line

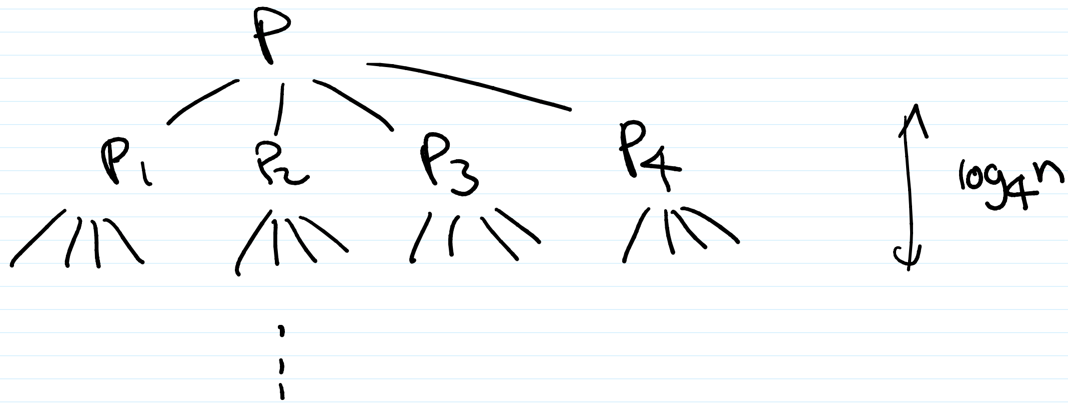


take ham-sandwich cut of left subset
right subset

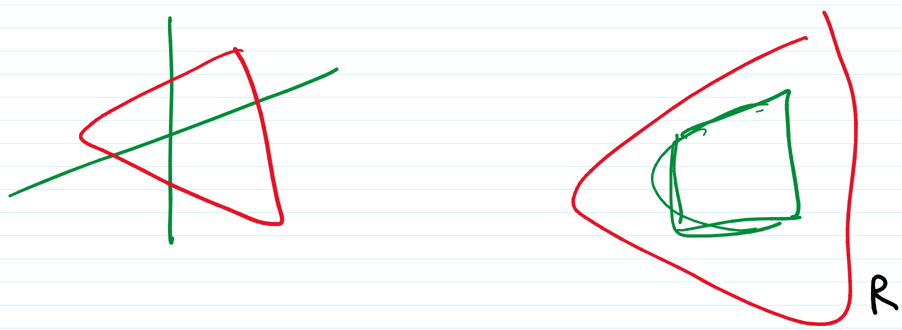
$\square$

Preprocessing:

just apply thm recursively!



Query:



$$Q(n) = O(\#\text{ cells intersecting } \partial R)$$

suffices to bound # cells intersecting a line

$$f(n) = 3 f\left(\frac{n}{4}\right) + O(1)$$

$$\Rightarrow O\left(n^{\log_4 3}\right)$$

$$\le \boxed{O\left(n^{0.793}\right)}$$ + k for report. query time

$\boxed{O(n)}$ space

⋮

Better? $O(n^{0.695})$ Edelsbrunner–Welzl '86

Better?

$O(n^{0.695})$ Edelsbrunner-Welzl '86

$O(n^{0.667})$ Haussler-Welzl '87

$\vdots$

Matoušek's Partition Thm '93 Fix any const r.

Can partition P into r subsets of size $\frac{n}{r}$ s.t.

(i) each subset is contained in a region

(ii) any line crosses $\leq \underline{C\sqrt{r}}$ regions

$$\Rightarrow f(n) = \underline{C\sqrt{r}}\, f\left(\frac{n}{r}\right) + O(1)$$

$$\Rightarrow O\left(n^{\frac{\log C\sqrt{r}}{\log r}}\right)$$

$$= O\left(n^{\frac{\log C + \frac{1}{2}\log r}{\log r}}\right)$$

$$= O\left(n^{\frac{1}{2} + \frac{\log C}{\log r}}\right)$$

$$= O\left(n^{\frac{1}{2}+\varepsilon}\right).$$

$\Rightarrow$ $O(n)$ space

$O(n^{\frac{1}{2}+\varepsilon})$ query time

$n^{1-\frac{1}{d}+\varepsilon}$ in $\mathbb{R}^d$.