October 15, 2024 3:18 PM

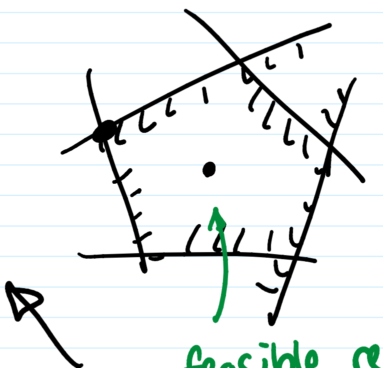
Linear Programming (LP)

optimization problem involving linear constraints

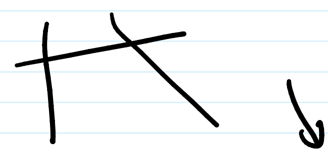
$$\begin{aligned} \min/\max \quad & c_1 x_1 + c_2 x_2 + \dots + c_d x_d \\ \text{s.t.} \quad & a_{11} x_1 + a_{12} x_2 + \dots + a_{1d} x_d \le b_1 \\ & \vdots \\ & a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nd} x_d \le b_n \end{aligned}$$

over vars $x_1, \dots, x_d \in \mathbb{R}$.

geometric interpretation:



intersection of n halfspaces in $\mathbb{R}^d$

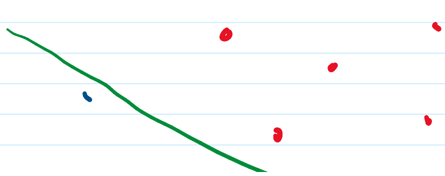


Rmk - from other courses, simplex method, ellipsoid method, interior-point methods, ...

poly time (in bit complexity)

Our interest - n large, d very small!

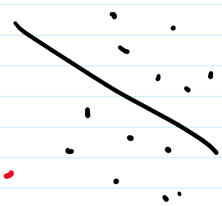
Ex1 Line Separation



Given $P^+ = \{ (a_1^+, b_1^+), \dots, (a_n^+, b_n^+) \}$

$P^- = \{ (a_1^-, b_1^-) \dots, (a_n^-, b_n^-) \}$.

find line separating P^+ & P^-.



$y = \xi x + \eta$ vars ξ, η.

$\Rightarrow$ min (don't care)

s.t. $b_i^+ \geq \xi a_i^+ + \eta$ $i = 1, .., n$

$b_i^- \leq \xi a_i^- + \eta$ $i = 1, .., n$

$a_i^+ \xi + \eta \leq b_i^+$

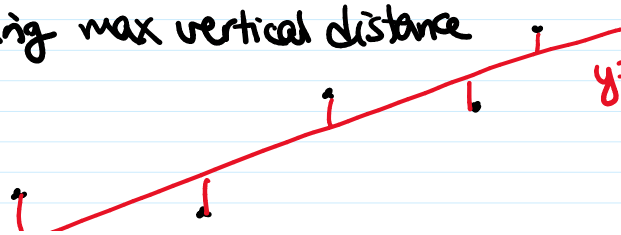
$a_i^- \xi + \eta \geq b_i^-$

over $\xi, \eta \in \mathbb{R}$

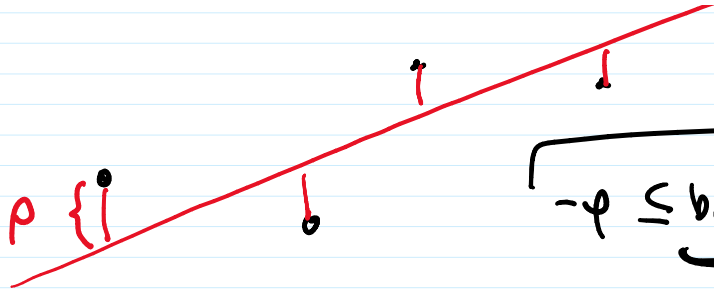
$\Rightarrow$ 2D LP

Ex2 Line fitting given $P = \{ (a_1, b_1) \dots, (a_n, b_n) \}$

find line minimizing max vertical distance



$y = \xi x + \eta$



$$\boxed{-\rho \le b_i - (\xi a_i + \eta) \le \rho}$$

min ρ

s.t.

$$-a_i\xi - \eta - \rho \le -b_i \quad \forall i=1,..,n$$
$$a_i\xi + \eta - \rho \le b_i \quad \forall i=1,..,n$$

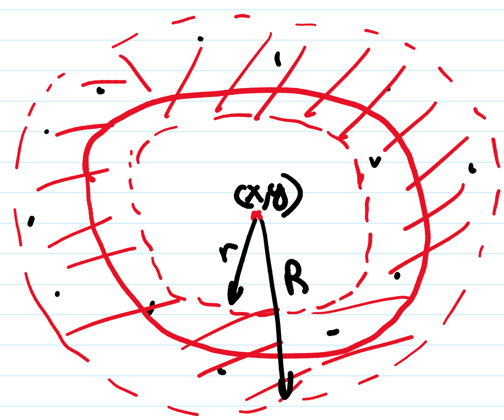
over $\xi, \eta, \rho \in \mathbb{R}$

$\Rightarrow$ 3D LP.

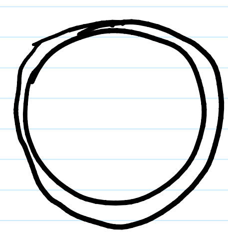
Ex3 (Circle fitting)

Given $P = \{(a_i, b_i)\}_{i=1}^{n}$

find min-area annulus enclosing P



annulus



min $\cancel{\pi}(R^2 - r^2)$

s.t.

$$(x-a_i)^2 + (y-b_i)^2 \le R^2 \quad i=1,..,n$$
$$(x-a_i)^2 + (y-b_i)^2 \ge r^2 \quad i=1,..,n$$

over $x, y, r, R \in \mathbb{R}$.

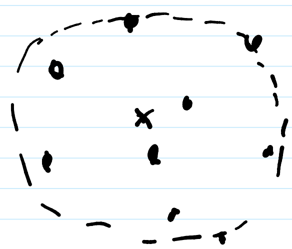
change of vars: $w = R^2 - x^2 - y^2$

$z = r^2 - x^2 - y^2$

$$\Rightarrow \begin{cases} \min\ w - z \\ \text{s.t. } -2a_i x - 2b_i y + a_i^2 + b_i^2 \leq w & i = 1, \dots, n \\ \quad -2a_i x - 2b_i y + a_i^2 + b_i^2 \geq z & i = 1, \dots, n \\ \quad \text{over } x, y, z, w \in \mathbb{R}. \end{cases}$$

$\Rightarrow$ 4D LP.

Ex4 smallest enclosing circle



$$\min\ r$$
$$\text{s.t. } (x - a_i)^2 + (y - b_i)^2 \leq r^2$$
$$x, y, r \in \mathbb{R}$$

$z = r^2 - x^2 - y^2$

$$\min\ x^2 + y^2 + z$$
$$\text{s.t. } -2a_i x - 2b_i y + a_i^2 + b_i^2 \leq z$$
$$x, y, z \in \mathbb{R}$$

not LP, but linear constraints

min convex obj fn.

$\Rightarrow$ convex prog. in 3D.

Trivial Algm for LP

- construct intersection of n halfspaces

$\xRightarrow{\text{dual of CH}}$ $O(n \log n)$ for $d = 2, 3$

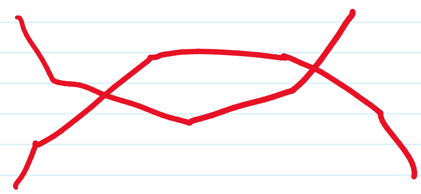
$\Rightarrow$ $O(n \log n)$ for $d = 2, 3$

worse for $d \geq 4$

if a feasible pt is given

- if a feasible pt not given,

  in 2D, compute lower envelope of lower halfspaces
  upper envelope of upper halfspace



& decide if they intersect

$O(n \log n)$ in 2D.

faster?

$O(n)$ time is possible in 2D!
(Megiddo '83 / Dyer '83)

idea - "prune-and-search"

try to remove a fraction of input

$\Rightarrow$ geometric series

$\Rightarrow$ $O(n)$ time

$n + \frac{3n}{4} + \left(\frac{3}{4}\right)^2 n + \cdots$