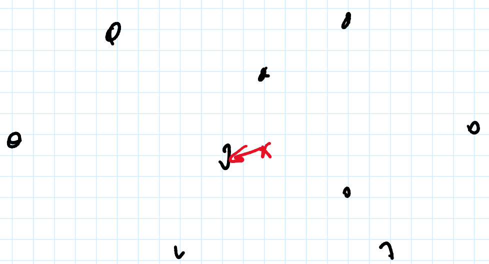
September 26, 2024 3:20 PM

Voronoi Diagrams

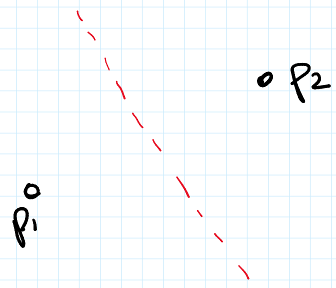
"Post Office" Problem — preprocess set P of n pts (called sites) $p_i = (a_i, b_i)$,

given query pt $q = (x, y)$, can find nearest site of q

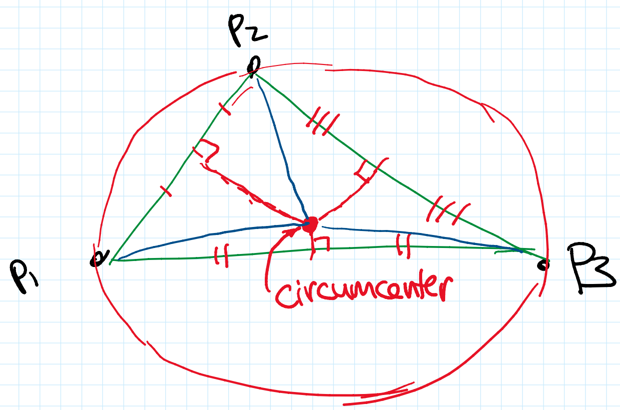


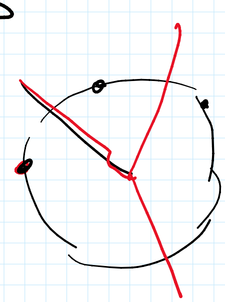
idea - create a "map" of all sol'ns

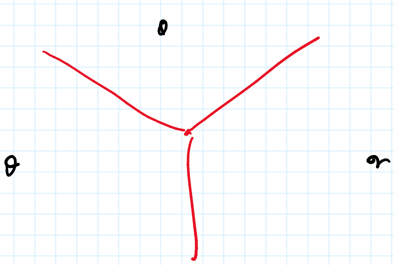
$n=2$



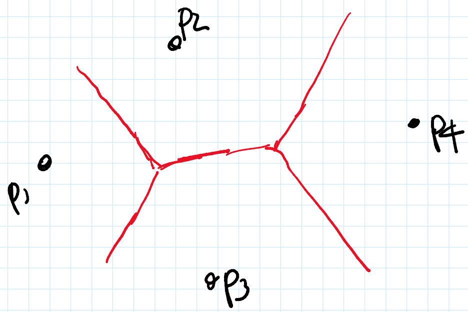
$n=3$

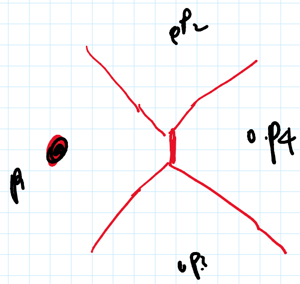


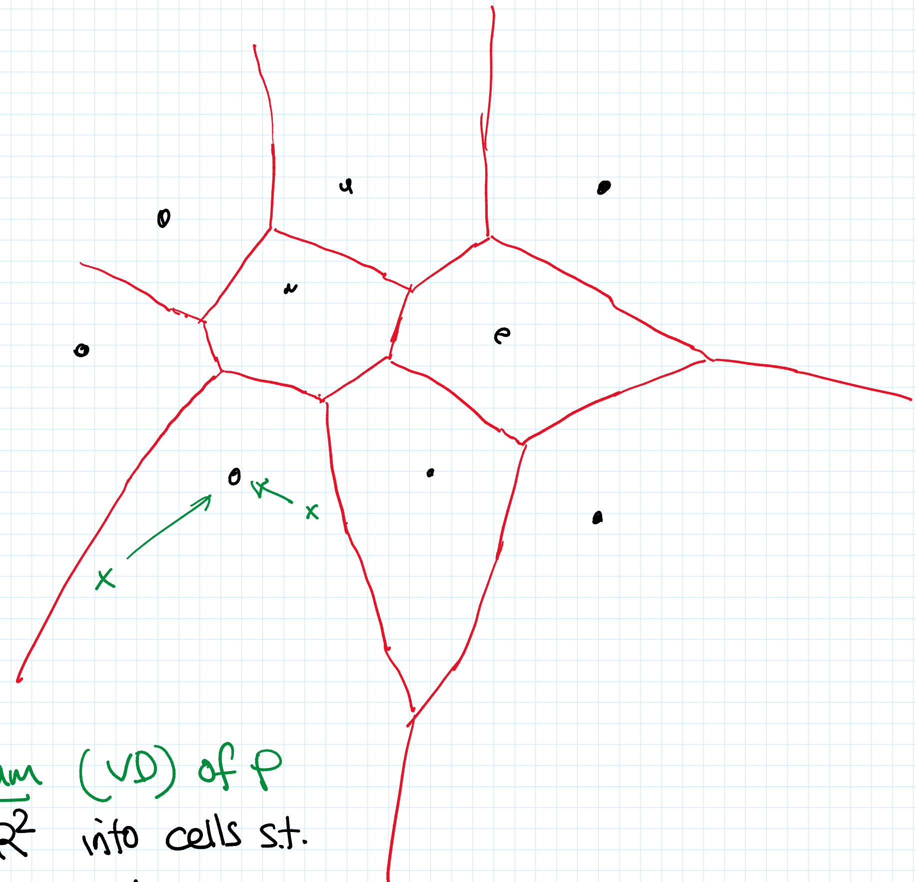




larger n?







Def The Voronoi diagram (VD) of P is subdivision of $\mathbb{R}^2$ into cells s.t.
pts in same cell have the same nearest site

$$V(p_i) = \{ q \in \mathbb{R}^2 : d(p_i, q) \leq d(p_j, q) \ \forall j \}$$

Sol'n to post office problem:
just look up which cell q is in
("point location")

History: Voronoi 1908
Dirichlet tessellation 1850
or Descartes 1644?

geography: "Thiessen polygons" 1911

natural sciences: "Blum's transform" 1967

crystallography - - -

CS: Shamos, Hoey 1975

$p_i = (a_i, b_i)$

Alternative view - given $q = (x, y)$,

want i to minimize

$$d(p_i, q)^2 = (x - a_i)^2 + (y - b_i)^2$$

$$= x^2 + y^2 - 2a_i x - 2b_i y + a_i^2 + b_i^2$$

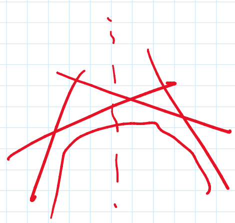
Same as minimizing $\underline{-2a_i x - 2b_i y + a_i^2 + b_i^2}$

$z = 3x - 4y + 5$

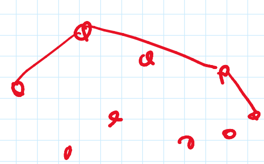
duality transformation

Def Given point $p = (a, b)$, define its lifted plane: $z = -2ax - 2by + a^2 + b^2$

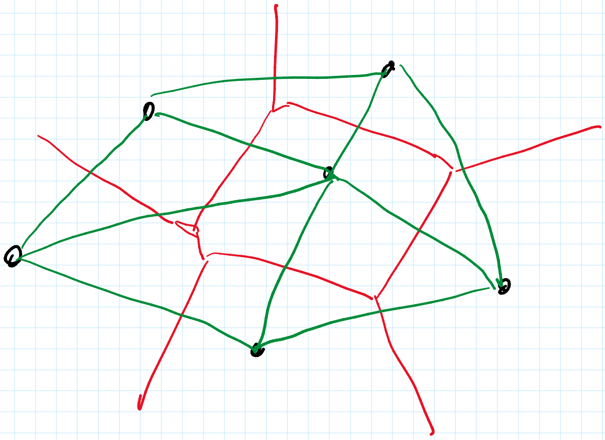
Obs VD in $\mathbb{R}^2$ $\Longleftrightarrow$ (proj of) lower envelope of lifted planes in $\mathbb{R}^3$

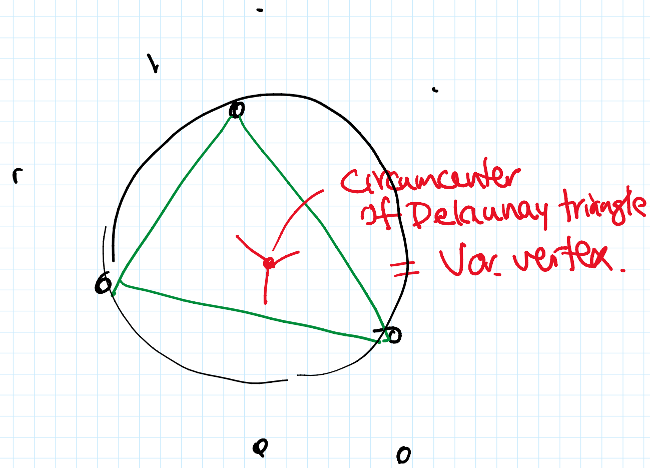


(dual point $(-2a, -2b, -a^2 - b^2)$)



Def The Delaunay triangulation (DT) of P is the planar-graph dual of the VD (with P as vertices)



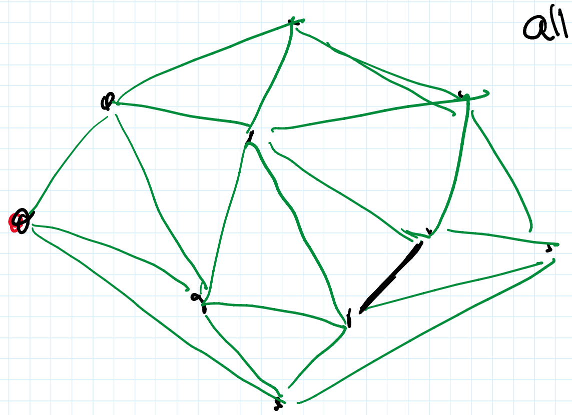


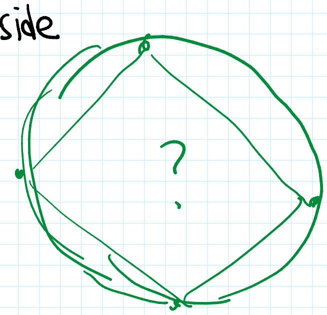
Properties:

- $p_i p_j p_k$ is a DT triangle iff
  all pts are outside circle thru p_i, p_j, p_k

called empty circle property

$p_i p_j$ is a DT edge iff
$\exists$ circle thru p_i, p_j s.t.
all pts are outside





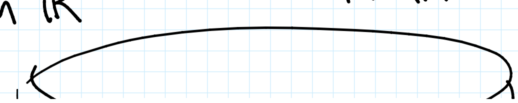
degenerate

By Obs, DT in $\mathbb{R}^2$ $\iff$ proj of upper hull of $\{(-2a_i, -2b_i, -a_i^2 - b_i^2)\}_{i=1}^n$

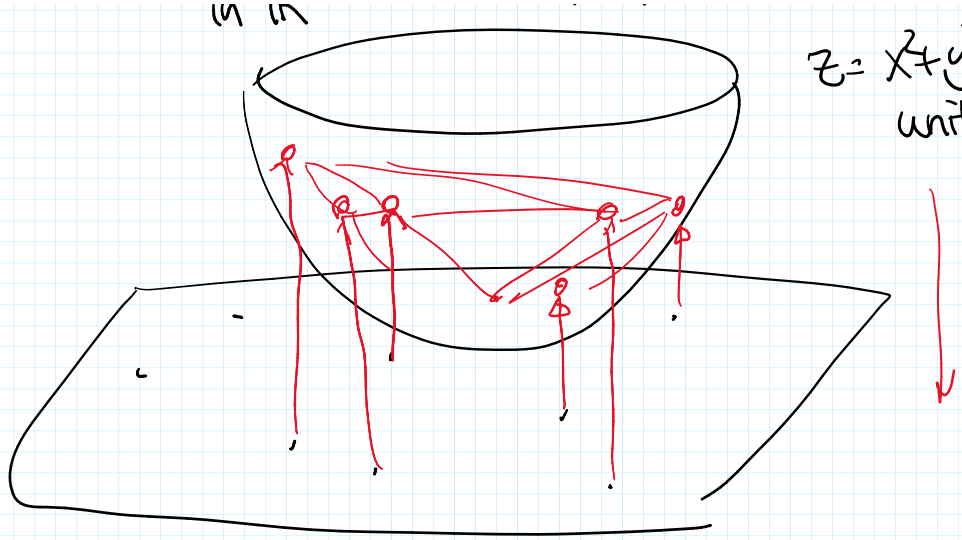
$\iff$ lower hull of $\{(a_i, b_i, a_i^2 + b_i^2)\}_{i=1}^n$
(pt set in $\mathbb{R}^3$)

lifting transform:

$p = (a, b) \longrightarrow \hat{p} = (a, b, a^2 + b^2)$
in $\mathbb{R}^2$ in $\mathbb{R}^3$



$z = x^2 + y^2$



$z = x^2 + y^2$

unit paraboloid

Consequences: p_l outside circle thru p_i, p_j, p_k

iff $\hat{p}_l$ above plane thru $\hat{p}_i, \hat{p}_j, \hat{p}_k$

iff
$$\begin{vmatrix} 1 & x_i & y_i & x_i^2+y_i^2 \\ 1 & x_j & y_j & x_j^2+y_j^2 \\ 1 & x_k & y_k & x_k^2+y_k^2 \\ 1 & x_l & y_l & x_l^2+y_l^2 \end{vmatrix} > 0$$

assuming p_i, p_j, p_k is ccw turn