$$= \frac{\text{total degree}}{n}$$

$$= \frac{2(\#\text{edges})}{n}$$

$$\leq \frac{2 \cdot 3n}{n} = 6.$$

$E[\text{Total}] \leq 6n.$ $\square$

by linearity of expectation

($T(n) \leq 6 + T(n-1)$).

how to do lines 2-3? "point location" problem

Clarkson-Shor / Mulmuley:

$O(\log n)$ expected time

by maintaining a search structure

$\Rightarrow$ randomized $\boxed{O(n \log n)}$ expected algm

(no bad input)

Algm 3: Divide & Conquer (Preparata, Hong '77)

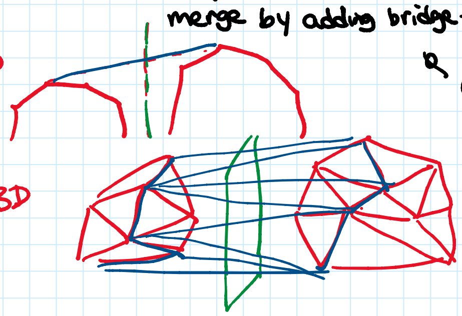
idea - sort in x first

compute left & right UH

merge by adding bridge face

2D

3D



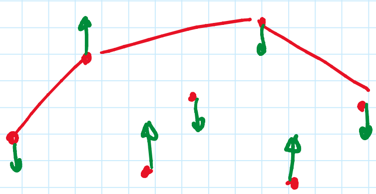
& doable in $O(n)$ time

$\Downarrow$

$O(n \log n)$ time

How to merge in $O(n)$ time?

"kinetic interpretation"



Obs Given $P = \{(x_i, y_i, z_i)\}_{i=1}^{n} \subset \mathbb{R}^3$,

define projection $\hat{P}(t) = \{(x_i, z_i - y_i t)\}_{i=1}^{n} \subset \mathbb{R}^2$

Then edges of $UH(P)$

$\Longleftrightarrow$ edges of $\boxed{UH(\hat{P}(t))}$ over all t

(Pf: $p_i p_j$ is an edge iff $\exists s, t, b$ s.t.

$z_i = s x_i + t y_i + b$ $\qquad$ $z_i - t y_i = s x_i + b$

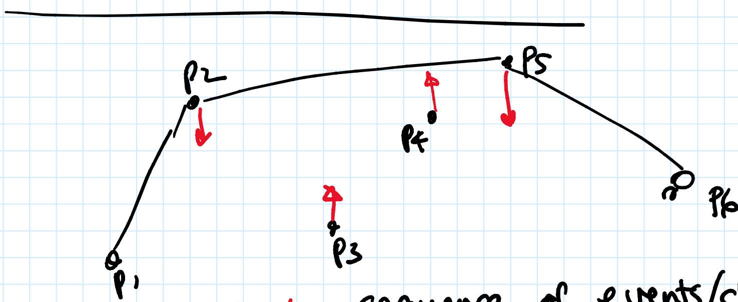
$z_j = s x_j + t y_j + b$ $\qquad$ $z_j - t y_j = s x_j + b$

(Rt: $p_i p_j$ is an edge in ...

$$z_i = sx_i + ty_i + b$$
$$z_j = sx_j + ty_j + b$$
$$\forall k,\ z_k < sx_k + ty_k + b.$$

$$z_i - ty_i = sx_i + b$$
$$z_j - ty_j = sx_j + b$$
$$z_k - ty_k < sx_k + b.$$
□

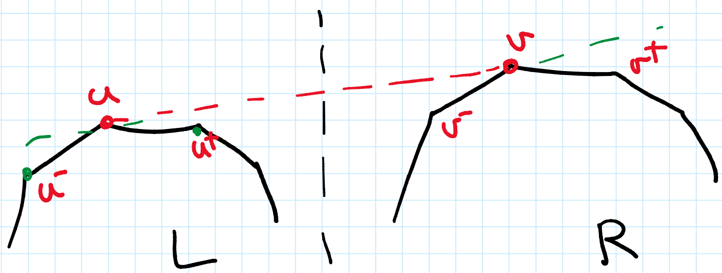
idea- think of t as time!
track how $UH(\hat{P}(t))$ changes over time

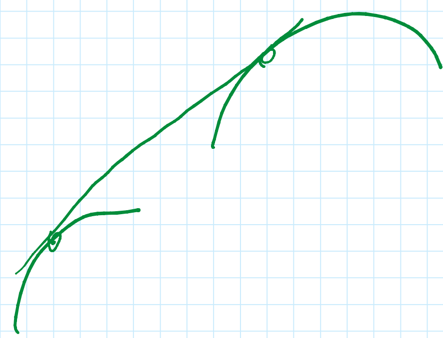


Output: sequence of events/changes

sorted by time

e.g. p_4 appears between p_2 & p_5
p_5 disappears
⋮





$A = \text{merge}(L, R)$:

let (u, v) be initial bridge at $t = -\infty$
repeat {
- t_1 = time for next event of L
- t_2 = time for next event of R
- t_3 = time when uu^+v collinear
- t_4 = time when u^-uv collinear
- t_5 = time when uvv^+ collinear
- t_6 = time when uv^-v collinear
- $t \leftarrow$ smallest among $t_1, \ldots, t_6$
- if $t = t_1$ then
  - change in L is left of u

if $t = t_1$ then
    if change in L is left of u
        copy change to A
if $t = t_2$ then
    if change in R is right of v
        copy change to A
if $t = t_3$ then
    add "u^+ appears between u and v" to A
    $u \Leftarrow u^+$
if $t = t_4$ then
    add "u disappears" to A
    $u \Leftarrow u^-$
if $t = t_5$ then
    add "v disappears" to A
    $v \Leftarrow v^+$
if $t = t_6$ then
    add "v^- appears between u & v" to A
    $v \Leftarrow v^-$

}

events $O(n)$

$\Rightarrow$ merge time $O(n)$

$\Rightarrow$ D&C $T(n) = 2T(\frac{n}{2}) + O(n)$

$\Rightarrow \boxed{O(n \log n)}$

Other Algms:

prune-divide & conquer: $O(n \log^2 h)$ [Edelsbrunner, Shi '91]

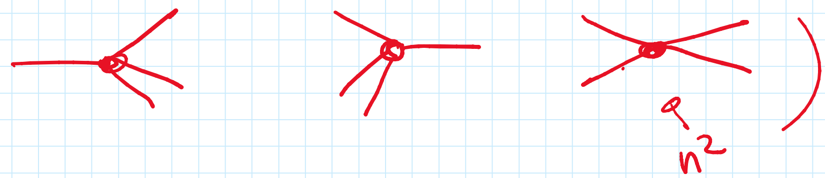
grouping: $O(n \log h)$ [C'95]

CH in $\mathbb{R}^d$ ($d \geq 4$):

combinatorics $O(\underline{n^{\lfloor d/2 \rfloor}})$ faces in worst case

(modify dual pf:

$d=4$:



)

algms:

gift-wrapping	$O(nf)$	$f = \#$CH faces
incremental	$O(n^{\lceil d/2 \rceil})$	worst-case (Seidel '81)
	$O(n^{\lfloor d/2 \rfloor})$	randomized (Clarkson-Shor '89)
derandomized	$O(\underline{n^{\lfloor d/2 \rfloor}})$	worst case (Chazelle '93)
dual sweep	$O(n^2 + f \log n)$	Seidel '86
C.-Snoeyink-Yap '95	$O((n+f)\log^2 f)$	for $d=4$
Amato-Ramos '96	$O((n+f)\log^3 f)$	for $d=5$
OPEN	$O((n+f)\log^c f)$	for $d \geq 6$??

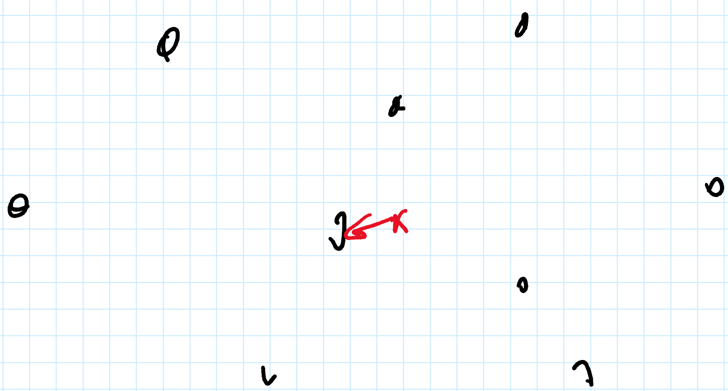
implementations: "qhull"
Clarkson's "hull"
CGAL (Comp. Geom. Algms Library)

called sites

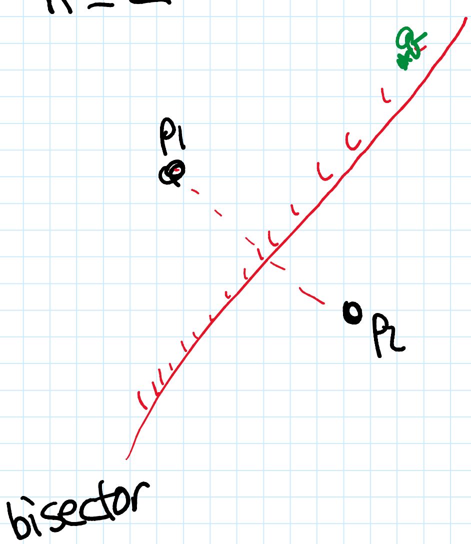
"Post Office" Problem   preprocess set P of n pts $p_i=(a_i,b_i)$,

given query pt $q=(x,y)$,

can find nearest site of q



e.g. $n=2$



$n=3$