Chapter 2
Convexity

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DREAM INTERPRETATION
Simplified. Everything’s either concave or convex, so whatever you dream will be something with sex.
– Piet Hein.

Version: 0.1

2.1. Some properties of convex sets

We here introduce three theorems about convex sets and convexity, which turns out to be quite useful.

Theorem 2.1.1 (Radon’s Theorem). Let \( P \subseteq \mathbb{R}^d \) be a set of \( n \geq d + 2 \) points in \( \mathbb{R}^d \). Then one can partition \( P \) into two disjoint sets \( X, Y \), such that \( P = X \cup Y \), \( X \cup Y = \emptyset \), and \( \text{CH}(X) \cap \text{CH}(Y) \neq \emptyset \).

Proof: Proof by drawing of the 2d case. Higher dimension is proved in Section 2.2.

Theorem 2.1.2 (Carathéodory’s Theorem). Let \( P \) be a set of \( n \) points in \( \mathbb{R}^d \), and let \( p \in \text{CH}(P) \) be an arbitrary point. Then \( P \) can be written as the convex combination of \( d + 1 \) points of \( P \).

Proof: We prove the 2d case. Let \( X = \text{CH}(P) \). Since \( X \) is a convex polygon, let \( v \) its bottom vertex. Connect \( v \) to all the other vertices of \( X \). This partitions \( X \) into triangles, and one of them contains \( p \). As such, the point \( p \) is in the convex hull of the three vertices of this triangle, which means that \( p \) can be written as a convex combination of only these three points.

The result for higher dimensions follows from Radon’s Theorem, and is omitted here.

2.1.1. Helly’s theorem

Lemma 2.1.3. Let \( F \) be a set of four convex sets \( S_1, S_2, S_3, S_4 \) in the plane, such that any three of them have a non-empty intersection. Then, all the convex sets have a non-empty intersections.

Proof: Let \( p_{-i} \) (or simply \( -i \)) denote any point that lies in \( \cap_{k \in \{1,2,3,4\} \setminus \{i\}} S_k \). The four points \( p_{-1}, p_{-2}, p_{-3}, p_{-4} \), by Radon’s theorem, can be decomposed into two sets \( X \) and \( Y \), such that \( \text{CH}(X) \cap \text{CH}(Y) \neq \emptyset \).

There are two possibilities:

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Claim 2.2.1. Let \( P = \{p_1, \ldots, p_{d+2}\} \) be a set of \( d+2 \) points in \( \mathbb{R}^d \). There are real numbers \( \beta_1, \ldots, \beta_{d+2} \), not all of them zero, such that \( \sum \beta_i p_i = 0 \) and \( \sum \beta_i = 0 \).

Proof: Indeed, set \( q_i = (p_i, 1) \), for \( i = 1, \ldots, d+2 \). Now, the points \( q_1, \ldots, q_{d+2} \in \mathbb{R}^{d+1} \) are linearly dependent, and there are coefficients \( \beta_1, \ldots, \beta_{d+2} \), not all of them zero, such that \( \sum_{i=1}^{d+2} \beta_i q_i = 0 \). Considering only the first \( d \) coordinates of these points implies that \( \sum_{i=1}^{d+2} \beta_i p_i = 0 \). Similarly, by considering only the \((d+1)\)st coordinate of these points, we have that \( \sum_{i=1}^{d+2} \beta_i = 0 \).

Theorem 2.2.2 (Radon’s theorem). Let \( P = \{p_1, \ldots, p_{d+2}\} \) be a set of \( d+2 \) points in \( \mathbb{R}^d \). Then, there exist two disjoint subsets \( C \) and \( D \) of \( P \), such that \( \mathcal{CH}(C) \cap \mathcal{CH}(D) \neq \emptyset \) and \( C \cup D = P \).
Proof: By Claim 2.2.1 there are real numbers $\beta_1, \ldots, \beta_{d+2}$, not all of them zero, such that $\sum_i \beta_i p_i = 0$ and $\sum_i \beta_i = 0$.

Assume, for the sake of simplicity of exposition, that $\beta_1, \ldots, \beta_k \geq 0$ and $\beta_{k+1}, \ldots, \beta_{d+2} < 0$. Furthermore, let $\mu = \sum_{i=1}^{k} \beta_i = -\sum_{i=k+1}^{d+2} \beta_i$. We have that

$$\sum_{i=1}^{k} \beta_i p_i = -\sum_{i=k+1}^{d+2} \beta_i p_i.$$ 

In particular, $v = \sum_{i=1}^{k} (\beta_i/\mu) p_i$ is a point in $C\mathcal{H}(\{p_1, \ldots, p_k\})$. Furthermore, for the same point $v$ we have $v = \sum_{i=k+1}^{d+2} -((\beta_i/\mu)) p_i \in C\mathcal{H}(\{p_{k+1}, \ldots, p_{d+2}\})$. We conclude that $v$ is in the intersection of the two convex hulls, as required. $\blacksquare$

2.3. Bibliographical notes

The material here is pretty standard – see Chapter 1 in Matoušek [Mat02].

References