## Chapter 1

## Introduction and Convex Hulls

By Sariel Har-Peled, January 26, $2023{ }^{(1)}$

The events of 8 September prompted Foch to draft the later legendary signal: "My centre is giving way, my right is in retreat, situation excellent. I attack." It was probably never sent.

-     - The first world war, John Keegan..

Version: 0.1

### 1.1. Introduction

(I) Administrivia.
(A) https://courses.engr.illinois.edu/cs498sh3/sp2023/.
(B) Book: The four marks.
(C) At least one midterm.
(D) Weekly homeworks.
(II) Examples:
(A) Convex-hulls. Mixing things.
(B) Robot motion planning. Shortest path.
(C) Where am I anyway? Voronoi diagrams?
(D) What am I seeing anyway? Ray shooting.
(E) Will I get flooded? GIS.
(F) Self driving cars.
(G) CAD/CAM.

### 1.2. Convex hull

Let $P=\left\{p_{1}, \ldots, p_{n}\right\}$ be a set of points (say in the plane). The span of $P$ is the linear subspace

$$
\operatorname{span}(P)=\left\{\sum_{i=1}^{n} \alpha_{i} p_{i} \mid \alpha_{1}, \ldots, \alpha_{n} \in \mathbb{R}\right\} .
$$

The affine hull of a set of points in the plane is the set

$$
\operatorname{affine-hull}(P)=\left\{\sum_{i=1}^{n} \alpha_{i} p_{i} \mid \alpha_{1}, \ldots, \alpha_{n} \in \mathbb{R}, \text { and } \sum_{i=1}^{n} \alpha_{i}=1\right\} .
$$

[^0]The affine hull of two points in the plane is the line that passes through them. The affine hull of three points in the plane in general position is the whole plane.

The convex-hull of $P$ is the set

$$
\mathcal{C H}(P)=\left\{\sum_{i=1}^{n} \alpha_{i} p_{i} \mid \alpha_{1}, \ldots, \alpha_{n} \geq 0, \text { and } \sum_{i=1}^{n} \alpha_{i}=1\right\} .
$$

The convex-hull of two points in the plane is the close segment connecting them. The convex-hull of three points in the plane is the close triangle segment they form. But what is the convex-hull of four points, or more?

A set $X \subseteq \mathbb{R}^{d}$ is convex, if for any two points $p, q \in X$, we have that the segment $p q \subseteq X$, where $p q=C \mathcal{H}(\{p, q\})$.

Claim 1.2.1. For any set $P \subseteq \mathbb{R}^{d}$, the set $X=C \mathcal{H}(P)$ is convex.
Proof: Let $P=\left\{p_{1}, \ldots, p_{n}\right\}$, and let $x=\sum_{i=1}^{n} \alpha_{i} p_{i}$ and $y=\sum_{i=1}^{n} \beta_{i} p_{i}$ any two points in $X$. That is $\alpha_{1}, \ldots, \alpha_{n} \geq 0, \sum_{i} \alpha_{i}=1, \beta_{1}, \ldots, \beta_{n} \geq 0$, and $\sum_{i} \beta_{i}=1$. Let $\gamma_{i}(t)=t \alpha_{i}+(1-t) \beta_{i}$, and consider the point $z(t)=\sum_{i} \gamma_{i}(t) p_{i}$. Clearly, for all $t \in[0,1]$, we have

$$
\gamma_{i}(t) \geq 0 \quad \text { and } \quad \sum_{i=1}^{n} \gamma_{i}(t)=\sum_{i}\left(t \alpha_{i}+(1-t) \beta_{i}\right)=t \sum_{i} \alpha_{i}+(1-t) \sum_{i} \beta_{i}=t+(1-t)=1
$$

This implies that $z(t) \in \mathcal{C H}(P)$. Namely, for any $t \in[0,1]$, the point $z(t) \in x y=C \mathcal{H}(x, y)$ has the property that $z(t) \in \mathcal{C H}(P)$.

### 1.2.1. Convex hull in the plane

What is the convex-hull of a set of $P$ points in the plane? Consider its boundary - it is formed by segments of $P$. Clearly if for $p \in P$, we have that $p \in C \mathcal{H}(P-p)$, then $\mathcal{C H}(P)=C \mathcal{H}(P-p)$, where $P-p=P \backslash\{p\}$.

A point $p \in P$ is a vertex of $\mathcal{C H}(P)$ if $\mathcal{C H}(P) \neq \mathcal{C H}(P-p)$. Informally, it is a corner of the convex-hull of $P$.

Claim 1.2.2. If $p$ is a vertex of $\mathcal{C H}(P) \Longleftrightarrow$ there exists a line that separates $p$ from $P-p$.
So if you traverse the convex-hull of $P$ (say counterclockwise), then it is a close sequence of vertices and segments connecting them. Such a region is a polygon. In our case it is a convex polygon.

Q; How to compute the convex-hull?

### 1.2.2. Algorithms for the convex-hull in two dimensions

The input is a set of $P$ of $n$ points in the plane. The output is a convex polygon $C$ that its vertices are from $P$. Specifically, $C$ is represented as a circular list of its vertices, say, counterclockwise as we traverse the vertices. Thus, if the input is specified an array $P[1 \ldots n]$, the output is a sequence of integers $i_{1}, i_{2}, \ldots, i_{h}$ (i.e., $13,7, \ldots$ ), where the polygon forming the convex hull have the vertices $P\left[i_{1}\right], P\left[i_{2}\right], \ldots, P\left[i_{k}\right]$.

We start with the gift wrapping algorithm for computing convex-hull.

Lemma 1.2.3 (Jarvis march). Given a set $P$ of $n$ points in the plane, one can compute its convex-hull in $O(n h)$ time, where $h$ is the number of vertices of the convex-hull.

Proof: Starting with the point with minimum $x$-coordinate, repeatedly find the clockwise-most neighbor, and use it as the next edge in the convex-hull. Repeat this process till you get back to the original starting point. Since finding the clockwise point takes $O(n)$ time, this takes $O(n h)$ overall.

The above result is disappointing as the convex-hull might have many vertices (i.e., $h$ might be equal to $n$ ).

Lemma 1.2.4 (Graham scan). Given a set $P$ of $n$ points in the plane, one can compute its convex-hull in $O\left(n+T_{\text {sort }}(n)\right)$ time, where $T_{\text {sort }}(n)$ is the time to sort $n$ numbers.

Proof: We sort the points from left to right by their $x$-coordinates (by general position assumption, all points have distinct $x$-coordinates values). Let $P=\left\langle p_{1}, \ldots, p_{n}\right\rangle$ be the points in this sorted order. Let $L_{i}$ be the lower convex chain of $C_{i}=C \mathcal{H}\left(\left\{p_{1}, \ldots, p_{i}\right\}\right)$ - that is it is the polygonal curve starting at $p_{1}$ and ending at $p_{i}$, including only vertices of $C_{i}$ that form the bottom part of $C_{i}$. Assume that $L_{i}$ is stored in a stack $S$. We now handle the point $p_{i+1}$. The stack $S$ have $n_{i}=\left|L_{i}\right|$ points before handling $p_{i+1}$. If $S[$ top -1$] \rightarrow S[$ top $] \rightarrow p_{i+1}$ form a right turn, then $S[$ top $]$ can not be a vertex of $C_{i+1}$. We pop it from the top of $S$. We repeat this process till $S$ [top -1] $\rightarrow S$ [top] $\rightarrow p_{i+1}$ form a left turn. The algorithm then pushes $p_{i+1}$ to $S$, and moves to the next point.

The algorithm then repeats the above process to compute the upper chain of the convex-hull.
The correctness is hopefully clear. As for running time, we charge every iteration that performs a pop, to the point being thrown away. Since every point can be thrown away at most once, it follows that the running time of the above algorithm is linear, ignoring the initial sorting.

Graham scan also works if you sort the points radially around one of the input points. Then one can compute the whole convex-hull in one go, instead of doing two different scans for the lower/upper chains.

### 1.2.2.1. Equivalence to sorting

Lemma 1.2.5. If the convex-hull of $n$ points in the plane can be computed in $T_{\mathrm{ch}}(n)$, then one can sort $n$ real numbers in $O\left(n+T_{\text {ch }}(n)\right)$.

Proof: Given number $x_{1}, \ldots, x_{n}$, lift the numbers to points $P=\left\{p_{1}, \ldots, p_{n}\right\}$, where $p_{i}=\left(x_{i}, x_{i}^{2}\right)$. Clearly, the convex-hull $C=C \mathcal{H}(P)$ would include all the points of $P$, as the lifted points lie on the convex curve formed by the parabola $y=x^{2}$. The order in which the vertices of $C$ are listed in the output polygon is the same order of the corresponding numbers $x_{1}, \ldots, x_{n}$ in sorted order, up to maybe rotation of the circular list (and maybe reversing it if one wants increasing order). Clearly, extracting the sorted numbers from $C$ can be done in $O(n)$ time.

It follows that sorting and computing the convex-hull of $n$ points in the plane, are equivalent as far as running time. In particular, since sorting (in general) takes $\Omega(n \log n)$ time, it follows that the convex-hull can not be computed faster.

### 1.3. Some properties of convex sets

Theorem 1.3.1 (Radon's Theorem). Let $P \subseteq \mathbb{R}^{d}$ be a set of $n \geq d+2$ points in $\mathbb{R}^{d}$. Then one can partition $P$ into to two disjoint sets $X, Y$, such that $P=X \cup Y, X \cup Y=\emptyset$, and $C \mathcal{H}(X) \cap C \mathcal{H}(Y) \neq \emptyset$.

Proof: Proof by drawing of the 2d case. Higher dimension is somewhat harder and we omit it here.
Theorem 1.3.2 (Carathéodory's Theorem). Let $P$ be a set of $n$ points in $\mathbb{R}^{d}$, and let $p \in C \mathcal{H}(P)$ be an arbitrary point. Then $P$ can be written as the convex combination of $d+1$ points of $P$.

Proof: Proof by drawing of the triangulation of 2d convex-hull. Higher dimension follows from Radon's Theorem, and is omitted here.

### 1.3.1. Helly's theorem

Lemma 1.3.3. Let $\mathcal{F}$ be a set of four convex sets $S_{1}, S_{2}, S_{3}, S_{4}$ in the plane, such that any three of them have a non-empty intersection. Then, all the convex sets have a non-empty intersections.

Proof: Let $p_{-i}$ (or simply $-i$ ) denote any point that lies in $\cap_{k \in\{1,2,3,4\}-i} S_{k}$. The four points $p_{-1}, p_{-2}, p_{-3}, p_{-4}$, by Radon's theorem, can be decomposed into two sets $X$ and $Y$, such that $C \mathcal{H}(X) \cap C \mathcal{H}(Y) \neq \emptyset$.


Figure 1.1
There are two possibilities:
(A) $|X|=|Y|=2$. For concreteness, let $X=p_{-1}, p_{-3}$ and $X=p_{-2}, p_{-4}$. Observe that $p_{-1} \in S_{2}$ and $p_{-3} \in S_{2}$. So the segment $p_{-1} p_{-3}=\mathcal{C H}(X) \subseteq S_{2}$, by the convexity of $S_{2}$. Similarly, $C \mathcal{H}(X) \subseteq S_{4}$. Namely, $\mathcal{C H}(X) \subseteq S_{2} \cap S_{4}$. The same argument implies that $\mathcal{C H}(Y) \subseteq S_{1} \cap S_{3}$.
We conclude that $\mathcal{C H}(X) \cap \mathcal{C H}(Y)$ (which is not empty) is contained in $S_{1} \cap S_{2} \cap S_{3} \cap S_{4}$, as claimed.
(B) $X=\{-1,-2,-4\}$ and $Y=\{-3\}$. But then $S_{3}$ must contain the triangle $\Delta p_{-1} p_{-2} p_{-4}$, which in turn implies that $p_{-3} \in \Delta p_{-1} p_{-2} p_{-4} \subseteq S_{3}$. Namely, $p_{-3}$ is contained in all the four convex objects.

With an easy trick (which one?), this leads to the following.
Lemma 1.3.4 (Helly's theorem in the plane). Let $\mathcal{F}$ be a set of $n \geq 4$ convex sets in the plane, such that any three of them have a non-empty intersection. Then, all the convex sets in $\mathcal{F}$ have $a$ non-empty intersections.

More generally, the following is correct.
Theorem 1.3.5 (Helly's Theorem). Let $\mathcal{F}$ be a set of $n \geq d+2$ convex sets in $\mathbb{R}^{d}$, such that any $d+1$ of them have a non-empty intersection. Then, all the convex sets in $\mathcal{F}$ have a non-empty intersections.

### 1.3.2. Proof of Radon's Theorem in higher dimensions

Claim 1.3.6. Let $P=\left\{p_{1}, \ldots, p_{d+2}\right\}$ be a set of $d+2$ points in $\mathbb{R}^{d}$. There are real numbers $\beta_{1}, \ldots, \beta_{d+2}$, not all of them zero, such that $\sum_{i} \beta_{i} p_{i}=0$ and $\sum_{i} \beta_{i}=0$.

Proof: Indeed, set $q_{i}=\left(p_{i}, 1\right)$, for $i=1, \ldots, d+2$. Now, the points $q_{1}, \ldots, q_{d+2} \in \mathbb{R}^{d+1}$ are linearly dependent, and there are coefficients $\beta_{1}, \ldots, \beta_{d+2}$, not all of them zero, such that $\sum_{i=1}^{d+2} \beta_{i} q_{i}=0$. Considering only the first $d$ coordinates of these points implies that $\sum_{i=1}^{d+2} \beta_{i} p_{i}=0$. Similarly, by considering only the $(d+1)$ st coordinate of these points, we have that $\sum_{i=1}^{d+2} \beta_{i}=0$.

Theorem 1.3.7 (Radon's theorem). Let $P=\left\{p_{1}, \ldots, p_{d+2}\right\}$ be a set of $d+2$ points in $\mathbb{R}^{d}$. Then, there exist two disjoint subsets $C$ and $D$ of $P$, such that $\mathcal{C H}(C) \cap C \mathcal{H}(D) \neq \emptyset$ and $C \cup D=P$.

Proof: By Claim 1.3.6 there are real numbers $\beta_{1}, \ldots, \beta_{d+2}$, not all of them zero, such that $\sum_{i} \beta_{i} p_{i}=0$ and $\sum_{i} \beta_{i}=0$.

Assume, for the sake of simplicity of exposition, that $\beta_{1}, \ldots, \beta_{k} \geq 0$ and $\beta_{k+1}, \ldots, \beta_{d+2}<0$. Furthermore, let $\mu=\sum_{i=1}^{k} \beta_{i}=-\sum_{i=k+1}^{d+2} \beta_{i}$. We have that

$$
\sum_{i=1}^{k} \beta_{i} p_{i}=-\sum_{i=k+1}^{d+2} \beta_{i} p_{i}
$$

In particular, $v=\sum_{i=1}^{k}\left(\beta_{i} / \mu\right) p_{i}$ is a point in $\mathcal{C H}\left(\left\{p_{1}, \ldots, p_{k}\right\}\right)$. Furthermore, for the same point $v$ we have $v=\sum_{i=k+1}^{d+2}-\left(\beta_{i} / \mu\right) p_{i} \in \mathcal{C H}\left(\left\{p_{k+1}, \ldots, p_{d+2}\right\}\right)$. We conclude that $v$ is in the intersection of the two convex hulls, as required.


[^0]:    ${ }^{(1)}$ This work is licensed under the Creative Commons Attribution-Noncommercial 3.0 License. To view a copy of this license, visit http://creativecommons.org/licenses/by-nc/3.0/ or send a letter to Creative Commons, 171 Second Street, Suite 300, San Francisco, California, 94105, USA.

