HW 9

Due on **Friday, April 21, 2023 at 9am CST**

CS 498CG: Discrete & Computational Geometry, Spring 2023 Version: **2.1**

Instructions: As in previous homeworks.

---

24 (100 PTS.) Brunn-Minkowski and caps

Let $\mathbb{S}^{(n-1)}$ be the unit radius sphere in $\mathbb{R}^n$. For positive $t < 1/2$, the ***t-cap*** B is a set of all points on the sphere such that their x-coordinate (say) is $\geq t$. An upper bound on the measure of B follows readily from the theorem showing measure concentration on the sphere (the bound is $\mathbb{P}[B] \leq 2\exp(-nt^2/2)$) applying it to A, where A is the hemisphere made out of all points with negative x-coordinate. Prove an improved bound, by repeating the proof, but applying it to B and $-B$ instead.

25 (100 PTS.) All point set are lanky in high dimensions.

Let P be a set of n points in (say) $\mathbb{R}^n$ such that the diameter of P is at most 1. As a reminder, the ***diameter*** of P is $\mathrm{diam}(P) = \max_{p,q \in P} \|pq\|$. Let v be a random direction taken from the unit sphere in (say) $\lfloor n/4 \rfloor$ dimensions, and consider the ***projection width*** of P:

$$w_v(P) = \max_{p \in P} \langle p, v \rangle - \min_{p \in P} \langle p, v \rangle .$$

Geometrically, this is the distance between two hyperplanes enclosing P between them with their normal being v.

Prove that $\mathbb{P}[w_v(P) \geq f(n)] \leq 1/n^{20}$. Here, you should provide an explicit function $f(n)$ that is as small as possible as a function of n (e.g., the claim definitely holds for $f(n) = 1/\ln n$, for n sufficiently large, but one can do much better).

Conclude that the width of P is at most $f(n)$.

26 (100 PTS.) Many points, same distance in high dimensions

Consider picking uniformly and independently a set P of m points from the unit sphere $\mathbb{S}^{(n-1)} \subseteq \mathbb{R}^n$, where $\mathbb{S}^{(n-1)}$ is the unit radius sphere in $\mathbb{R}^n$. Let $\varepsilon \in (0, 1/10)$ be a fixed constant. Since the mass of the sphere concentrates near its equator, it is not hard to show that for all points $p, q \in P$, we have that $\|pq\| \in [\sqrt{2} - \varepsilon, \sqrt{2} + \varepsilon]$. Using only material seen in class, prove a lower bound (as large as possible) on the number m (as a function of n and ε) such that this property holds with probability $\geq 1/2$.