HW 8

Due on **Friday, March 31, 2023 at 9am CST**

CS 498CG: Discrete & Computational Geometry, Spring 2023 Version: **1.1**

Instructions: As in previous homeworks.

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21 (100 PTS.) Geometric obesity is good

For a constant $\alpha \geq 1$, a planar convex region $\mathcal{o}$ is ***α-fat*** if the ratio $R(\mathcal{o})/r(\mathcal{o}) \leq \alpha$, where $R(\mathcal{o})$ and $r(\mathcal{o})$ are the radii of the smallest disk containing $\mathcal{o}$ and the largest disk contained in $\mathcal{o}$, respectively.

21.A. (50 PTS.) Prove that if a triangle $\triangle$ has all angles larger than β, then $\triangle$ is $\dfrac{1}{\sin(\beta/2)}$ fat. We will refer to such a triangle as being ***fat***.

21.B. (50 PTS.) Let $\mathcal{O}$ be a set of interior disjoint α-fat shapes all intersecting a common square $\square$. Furthermore, for every $\triangle \in \mathcal{O}$ we have that $\mathrm{diam}(\triangle) \geq c \cdot \mathrm{diam}(\square)$, where c is some constant. Prove that $|\mathcal{O}| = O(1)$. Here, the constant depends on c and α, try to figure out the right value/range of this constant as a function of α and c.

22 (100 PTS.) Quadtree for fat triangles

Let P be a triangular planar map of the unit square (i.e., each face is a triangle but it is not necessarily a triangulation), where all the triangles are fat and the total number of triangles is n.

22.A. (30 PTS.) Show how to build a compressed quadtree storing the triangles, such that every node of the quadtree stores only a constant number of triangles (the same triangle might be stored in several nodes) and given a query point p, the triangle containing p is stored somewhere along the point-location query path of p in the quadtree.

(Hint: Think about what is the right resolution to store each triangle.)

22.B. (30 PTS.) Show how to build a compressed quadtree for P that stores triangles only in the leaves, and such that every leaf contains only a constant number of triangles and the total size of the quadtree is $O(n)$.

Hint: Using the construction from the previous part, push down the triangles to the leaves, storing in every leaf all the triangles that intersect it. Use previous problem to argue that no leaf stores more than a constant number of triangles.

(Note, that in a compressed node the region it corresponds to is the set difference between two squares, and conceptually this regions is a "leaf" node of the quadtree.)

22.C. (10 PTS.) (Tricky but not hard) Show that one must use a compressed quadtree in the worst case if one wants linear space.

22.D. (30 PTS.) (Hard) We remind the reader that a ***triangulation*** is a triangular planar map that is compatible. That is, the intersection of triangles in the triangulation is either empty, a vertex of both triangles, or an edge of both triangles (this is also known as a ***simplicial complex***). Prove that a fat triangulation with n triangles can be stored in a regular (i.e., non-compressed) quadtree of size $O(n)$.

23 (100 pts.) Cell queries

Let $\widehat{\square}$ be a canonical grid cell. Given a compressed quadtree $\widehat{T}$, we would like to find the *single* node $v \in \widehat{T}$, such that $\mathsf{P} \cap \widehat{\square} = \mathsf{P}_v$. We will refer to such a query as a ***cell query***. Show how to support cell queries in a compressed quadtree in logarithmic time per query.