LECTURE 5 February 4^{th}, 2025

PART I Fundamental Concepts in Quantum Information

- One-qubit
  - superposition
  - measurement
  - unitary transformations
- Multiple qubits
  - superposition
  - partial measurements
  - quantum circuits
  - entanglement

TODAY Multi-qubit Systems, Quantum Circuits and Entanglement

Multi-qubit systems

Most common way of obtaining a qudit : 2 qubits

e.g. photon "0" = $\leftrightarrow$ or "1" = $\updownarrow$

state $\gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$

Say Alice has a qubit $|\psi\rangle$ and Bob has a qubit $|\phi\rangle \in \mathbb{C}^2$

Question 1 : What is the joint 4-d state ?

Question 2: If Bob applies a unitary $U \in \mathbb{C}^{2\times 2}$ to his qubit, what is the new 4-d state ?

Question 3 : If only Alice measures her qubit, what happens ?

Lets try to answer question 1.

We can view two qubits as a joint 4-d system :

$$\gamma_{00}|00\rangle + \gamma_{01}|01\rangle + \gamma_{10}|10\rangle + \gamma_{11}|11\rangle$$

Alice's qubit

Bob's qubit

Say Alice's qubit $|\psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$

Bob's qubit $|\phi\rangle = \beta_0|0\rangle + \beta_1|1\rangle$

Analogous to the probability rules for flipping two coins,

Overall amplitude of $|00\rangle = \alpha_0\beta_0$

So, $\gamma_{00} = \alpha_0\beta_0$, $\gamma_{01} = \alpha_0\beta_1$, $\gamma_{10} = \alpha_1\beta_0$, $\gamma_{11} = \alpha_1\beta_1$

This better be a quantum state. Let's check that

$$|\gamma_{00}|^2 + |\gamma_{01}|^2 + |\gamma_{10}|^2 + |\gamma_{11}|^2 = (|\alpha_0|^2 + |\alpha_1|^2)(|\beta_0|^2 + |\beta_1|^2) = 1 \cdot 1 = 1$$

More generally, what is the joint state of two qudits?

$\boxed{\text{QM Law 4}}$ Alice d-qudit Bob e-qudit

$$|\psi\rangle = \begin{array}{c} |1\rangle \\ \vdots \\ |d\rangle \end{array} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_d \end{bmatrix} \qquad |\phi\rangle = \begin{array}{c} |1\rangle \\ \vdots \\ |e\rangle \end{array} \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix}$$

Joint state is de-dimensional qudit

$$\begin{array}{c} |11\rangle \\ |12\rangle \\ \vdots \\ |1e\rangle \\ |21\rangle \\ \vdots \\ |de\rangle \end{array} \begin{bmatrix} \alpha_1\beta_1 \\ \alpha_1\beta_2 \\ \vdots \\ \alpha_1\beta_e \\ \alpha_2\beta_1 \\ \vdots \\ \alpha_d\beta_e \end{bmatrix} = \begin{bmatrix} \alpha_1 \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix} \\ \alpha_2 \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix} \\ \vdots \\ \alpha_d \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_e \end{bmatrix} \end{bmatrix} = |\psi\rangle \otimes |\phi\rangle$$

This operation is called a tensor product.

More generally, tensor product of two matrices A and B:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & \cdots & b_{1q} \\ \vdots & & \\ b_{p1} & \cdots & b_{pq} \end{bmatrix}$$

$m \times n$ matrix $\qquad\qquad$ $p \times q$ matrix

$$A \otimes B = \left[\begin{array}{c|c|c|c} a_{11} B & a_{12} B & \cdots & a_{1n} B \\ \hline & & & \boxed{} \\ \hline & & & a_{mn} B \end{array}\right]$$

Each block is a $p \times q$ matrix

$mp \times nq$ matrix

E.g. $|0\rangle \otimes |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \end{array} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} = \text{"}|00\rangle\text{"}$

$$|0\rangle \otimes |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|01\rangle$$

$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$$|+\rangle \otimes |0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

Demonstrates that tensor product is not a commutative operation

- $|+\rangle \otimes |-\rangle = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \otimes \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array}$

Let's do this in the ket notation

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle - \frac{1}{2}|11\rangle$$

Properties of tensor product

- Acts like "non-commutative multiplication"

$$(A+B)\otimes C = A\otimes C + B\otimes C$$

$$A\otimes(B+C) = A\otimes B + A\otimes C$$

$$A\otimes(B\otimes C) = (A\otimes B)\otimes C = A\otimes B\otimes C$$

E.g. Alice, Bob, Charlie. Joint state is

$$\begin{bmatrix}\alpha_0\\ \alpha_1\end{bmatrix} \quad \begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix} \quad \begin{bmatrix}\gamma_0\\ \gamma_1\end{bmatrix} \qquad \begin{array}{c} \begin{bmatrix}\alpha_0\beta_0\gamma_0\\ \alpha_0\beta_0\gamma_1\\ \vdots\\ \alpha_1\beta_1\gamma_1\end{bmatrix} \end{array} \begin{array}{l} |000\rangle\\ |001\rangle\\ \\ |111\rangle\end{array}$$

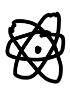
$$(A\otimes B)^{+} = A^{+}\otimes B^{+}$$

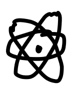
$$(A\otimes B)\cdot(C\otimes D) = (AC)\otimes(BD)$$

matrix multiplication

Quantum Circuits

Let's suppose Alice and Bob each prepared a qubit and got together

Alice 

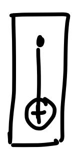
Bob 

These two particles have a joint state

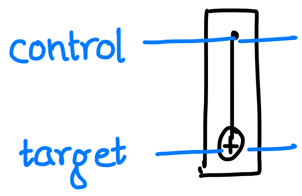
$$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$$

Alice Bob

Let's imagine they go in a physical device that changes their state (jointly)

e.g. the device applies a CNOT operation 

Definition (CNOT)



= 4×4 unitary transformation defined as follows

"If the control qubit is 0, do nothing
Else, apply a NOT to the target qubit"

Formally, $|00\rangle \to |00\rangle$ $\quad |10\rangle \to |11\rangle$
$|01\rangle \to |01\rangle$ $\quad |11\rangle \to |10\rangle$

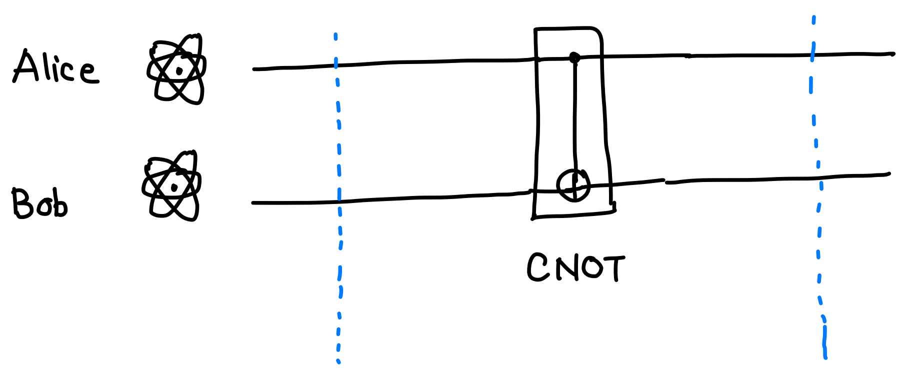
The matrix representation of CNOT is

$$\begin{array}{c} \\ |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} \longrightarrow \begin{bmatrix} \alpha_{00} \\ \alpha_{01} \\ \alpha_{11} \\ \alpha_{10} \end{bmatrix}$$

This is a permutation matrix, so it is easy to see that this is a unitary transformation

What is the joint state of Alice and Bob's qubit after CNOT ?

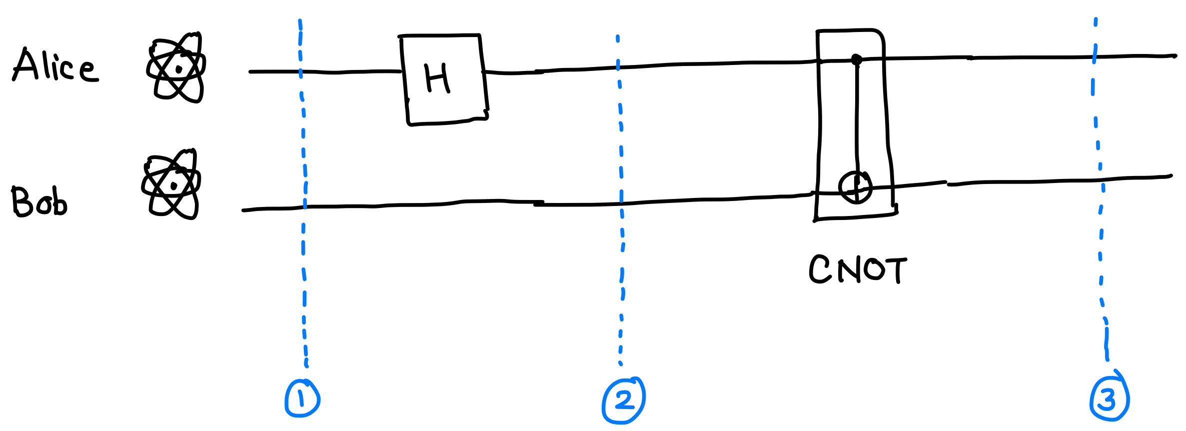
We draw this operation as a "quantum circuit" diagram



$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

$\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{11}|10\rangle + \alpha_{10}|11\rangle$

Let's draw a more interesting quantum circuit now



RECALL
$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$|0\rangle \to |+\rangle$
$|1\rangle \to |-\rangle$

What are the states at locations ①, ② and ③ ?

State at location 1 : $|00\rangle$

at location 2 : Alice only applies a gate to her qubit $H|0\rangle = |+\rangle$
so, state is

$$|+\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$$

at location 3 : CNOT swaps the amplitude of $|10\rangle$ & $|11\rangle$
so, state is

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle \longrightarrow \text{Bell state OR EPR pair}$$

$\boxed{\text{Theorem}}$ Bell state is not of the form $|\psi\rangle \otimes |\phi\rangle$ for any $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$

This means that such states can only arise when the particles interact

Proof Let $|\psi\rangle = \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$ and $|\phi\rangle = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

Then $|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$

Observe that the product of amplitudes on $|01\rangle$ & $|10\rangle$
= product of amplitudes on $|00\rangle$ & $|11\rangle$

for any tensor product state

This is not true for the Bell state ∎

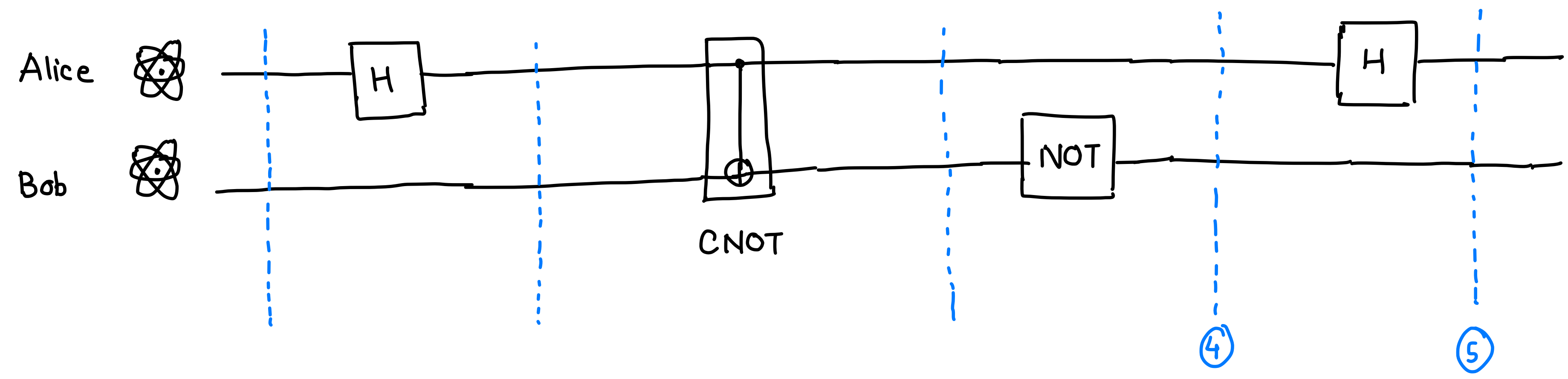
$\boxed{\text{Definition}}$ A state on multiple qubits is called entangled across a bipartition (of the qubits) if it cannot be written as $|\psi\rangle \otimes |\phi\rangle$ for any $|\psi\rangle \otimes |\phi\rangle$

It is not obvious just by looking at a state if its entangled

E.g. Is $\frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle + \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$ entangled?

$= |+\rangle \otimes |+\rangle$

Suppose Bob applies a NOT gate to her qubit → This is 2×2 unitary
How can we make sense of this ?



$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

At location 4 : $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

Suppose Alice now applies a H gate

At location 5 : with amplitude $\frac{1}{\sqrt{2}}$, state is $|\overset{\text{Alice}}{0}\overset{\text{Bob}}{1}\rangle$

applying H to Alice's qubit gives $(H|0\rangle)\otimes|1\rangle$

$$= |+\rangle \otimes |1\rangle$$
$$= \frac{1}{\sqrt{2}}|0\rangle\otimes|1\rangle + \frac{1}{\sqrt{2}}|1\rangle\otimes|1\rangle$$
$$= \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

with $\frac{1}{\sqrt{2}}$ amplitude, state is $|10\rangle$

applying H to Alice's qubit gives $(H|1\rangle)\otimes|0\rangle$

$$= |-\rangle \otimes |0\rangle$$
$$= \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle$$

Final state, $\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|10\rangle\right)$

$$= \frac{1}{2}|00\rangle + \frac{1}{2}|01\rangle - \frac{1}{2}|10\rangle + \frac{1}{2}|11\rangle$$

In general, say we have a 2-qubit state $\begin{bmatrix}\alpha_{00}\\ \alpha_{01}\\ \alpha_{10}\\ \alpha_{11}\end{bmatrix} \in \mathbb{C}^4$ and a 2×2 unitary $U = \begin{bmatrix} p & r \\ q & s\end{bmatrix}$

NEXT TIME What is the state after we apply U to 2^{nd} qubit ?