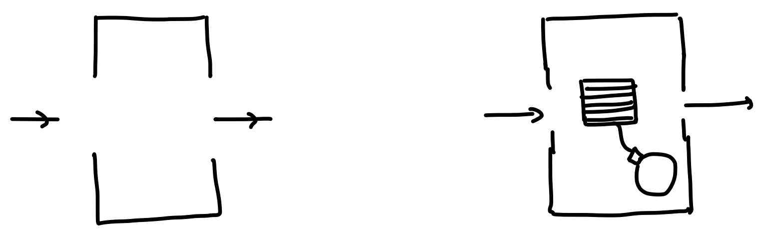
LECTURE 4 January 30^{th}, 2025

PART I Fundamental Concepts in Quantum Information
- superposition
- Measurements
- "Quantum Operations"

This lecture
- Elitzur-Vaidman Puzzle (contd)
- Unitary Transformations & Multi-qubit systems

RECAP Elitzur-Vaidman Bomb Tester

Dud OR Bomb fuse attached to "$\leftrightarrow$" filter



Nothing happens

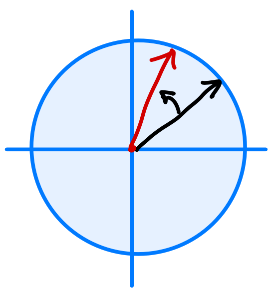
Photon measured in $\{\overset{\leftrightarrow}{|0\rangle}, \overset{\updownarrow}{|1\rangle}\}$ basis & bomb explodes if $|1\rangle$ is measured o/w photon comes out

Classically no chance of detecting
$|+\rangle$ state gave us 25% chance

Today we will give a better algorithm using new operations

Measurement gives us classical information and collapses the state
For quantum computing, we also need to be able to transform quantum states

Consider a qubit with real amplitudes



FACT For any θ, one can build a physical device that "rotates its state by θ"

E.g. by passing photon through a slab whose length depends on θ
or by shooting laser at an electron for time that depends on θ

The linear transformation that rotates by θ is given by the matrix

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

↳ where $|1\rangle$ goes

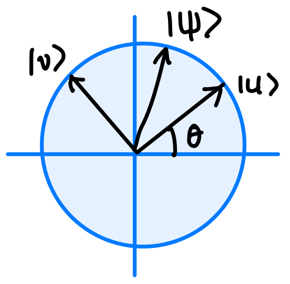
where $\begin{bmatrix}1\\0\end{bmatrix} = |0\rangle$ goes

Same operation works for complex amplitudes also

E.g. $\theta = 45^\circ$ $\quad R_{45^\circ} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ $\quad |0\rangle \longrightarrow |+\rangle$

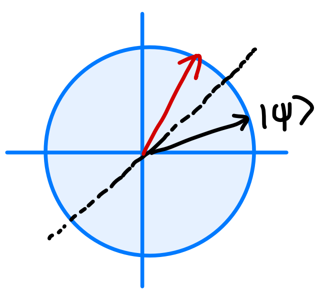
$|1\rangle \longrightarrow -|-\rangle$

Can simulate measurement in any basis with Rotation operations and Standard measurements



- Pass $|\psi\rangle$ through $R_{-\theta}$
  - $|u\rangle \longrightarrow |0\rangle$
  - $|v\rangle \longrightarrow |1\rangle$
- Standard Measurement
  - "$|0\rangle$" means measured "$|u\rangle$"
  - "$|1\rangle$" means measured "$|v\rangle$"
- Apply R_θ to the collapsed state
  - $|0\rangle \longrightarrow |u\rangle$
  - $|1\rangle \longrightarrow |v\rangle$

FACT Can also build a physical device that implements a reflection



E.g. if state was $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ & reflection thru 45°

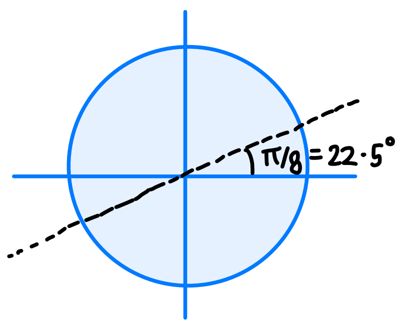
state becomes $\begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

The corresponding matrix is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ NOT gate

$|0\rangle \rightarrow |1\rangle$

$|1\rangle \rightarrow |0\rangle$

E.g.



sends $|0\rangle \rightarrow |+\rangle$ $\quad$ Matrix $\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ Hadamard gate H

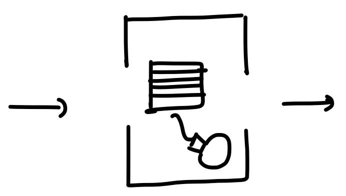
$|1\rangle \rightarrow |-\rangle$

E.g. (with complex amplitudes) Phase shift operation

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad S \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ i\beta \end{bmatrix}$$

valid qubit state

since $|\alpha|^2 + |i\beta|^2 = |\alpha|^2 + |\beta|^2 = 1$



- Start with $|0\rangle$
- Apply R_ε where $\varepsilon = \frac{\pi}{2n}$ for $n = 100000$
- Send into box
- If no explosion, repeat steps 2 and 3 n times
- Measure in standard basis

Case Dud : Qubit exits at angle ε

Case Bomb: $\mathbb{P}[\text{measure "}|0\rangle\text{"}] = (\cos\varepsilon)^2$ and then $|0\rangle$ exits

$\mathbb{P}[\text{measure "}|1\rangle\text{"}] = (\sin\varepsilon)^2 = \varepsilon^2$

If no explosion, photon comes out in state $|0\rangle$
Repeat steps 2 and 3 n times

Analyzing Full Algorithm

Case Dud : After n rotations, state of qubit is $|1\rangle$
since each rotation is $\frac{\pi}{2n}$

Case Bomb : Final state assuming no explosion is $|0\rangle$

$$\mathbb{P}[\text{explosion}] = n\cdot\varepsilon^2 = \frac{\pi^2}{4n} = \text{small}$$

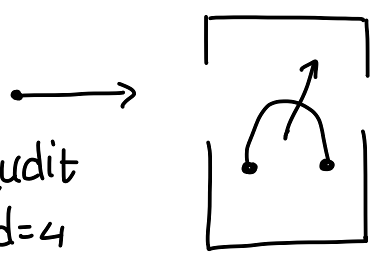
Measuring in standard basis : Dud $\longrightarrow |1\rangle$
Bomb $\longrightarrow |0\rangle$

Perfectly distinguish if there is no explosion

Rotation and Reflection operations are what are called unitary transformations! We will talk about them more generally so let us first introduce a qudit.

d-Qudit A quantum system in superposition of d basic states $|1\rangle, |2\rangle, \ldots\ldots |d\rangle$

$$\begin{pmatrix}\alpha_1 \\ \vdots \\ \alpha_d\end{pmatrix} = |\psi\rangle = \alpha_1|1\rangle + \ldots\ldots + \alpha_d|d\rangle \quad \text{where} \quad |\alpha_1|^2 + \cdots + |\alpha_d|^2 = 1$$



qudit
d=4

Measuring a qudit in the basis $\{|1\rangle, \ldots |d\rangle\}$

outcome is $|1\rangle$ with $|\alpha_1|^2$ & state collapses to $|1\rangle$
.... and so on

State of a qudit can also be changed by rotation/reflection in d-dimensions

QM law 3 A qudit state can be changed by any linear transformation that preserves length

These are called unitary transformations $\mathcal{U} \in \mathbb{C}^{d \times d}$

$$\mathcal{U} \text{ s.t. } \forall |\psi\rangle \quad \| \mathcal{U}|\psi\rangle \|^2 = \|\psi\|^2$$

$$\Leftrightarrow \quad (\mathcal{U}|\psi\rangle)^\dagger (U|\psi\rangle) = \langle\psi|\psi\rangle$$

$$\Leftrightarrow \quad \langle\psi| \underbrace{U^\dagger U}_{} |\psi\rangle = \langle\psi|\psi\rangle$$

This can only happen iff $U^\dagger U = I$

If $\mathcal{U} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_d \\ | & & | \end{pmatrix}$ then $\mathcal{U}^\dagger = \begin{pmatrix} — u_1^\dagger — \\ \vdots \\ — u_d^\dagger — \end{pmatrix}$

So, if $\mathcal{U}^\dagger \mathcal{U} = \begin{pmatrix} u_1^\dagger u_1 & u_1^\dagger u_2 & \cdots & u_1^\dagger u_d \\ u_2^\dagger u_1 & u_2^\dagger u_2 & \cdots & u_2^\dagger u_d \\ & \cdots & & \end{pmatrix} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

then columns of $\mathcal{U}$ form an orthonormal basis

Another equivalent defn: $\mathcal{U}\mathcal{U}^\dagger = I \Leftrightarrow$ inverse of $U = U^\dagger$

This implies that if U is allowed, then so is U^{-1}

All unitary operations are reversible

Another equivalent defn: $\mathcal{U}$ preseves angles (or inner products)

$$(\mathcal{U}|\phi\rangle)^\dagger \mathcal{U}|\psi\rangle = \langle\phi|\mathcal{U}^\dagger \mathcal{U}|\psi\rangle = \langle\phi|\psi\rangle$$

E.g. (On qubits) $R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ → check that it is unitary

$$NOT = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

E.g. (On qudits with $d=3$)

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{Permutation Matrix}} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} \gamma \\ \alpha \\ \beta \end{bmatrix}$$

preserves length

(Qudits with $d=4$)

$$\mathrm{SWAP} = \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 00 & 1 & 0 & 0 & 0 \\ 01 & 0 & 0 & 1 & 0 \\ 10 & 0 & 1 & 0 & 0 \\ 11 & 0 & 0 & 0 & 1 \end{array}$$

$$H^{\otimes 2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$$

Fun fact: Every unitary U has a square root

e.g. $\sqrt{R_\theta} = R_{\theta/2}$ and $\sqrt{\mathrm{NOT}} = \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$