LECTURE 2 Jan 23rd, 2025

PART I Fundamental Concepts in Quantum Information

This Lecture Understanding and Measuring one qubit

- Qubits and QM Law 1 (Superposition)
- Measurements and QM Law 2 (Born's Rule)
  - standard
  - in a different basis
- Uncertainty Principle
- Global vs Relative Phase

QM Law 1

RECALL Qubit quantum state in superposition of two basic states "$|0\rangle$" and "$|1\rangle$"

"α amplitude on $|0\rangle$, β amplitude on $|1\rangle$"
where α, β are complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$ $\quad \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$

E.g. a photon may have the state "$\frac{1}{\sqrt{2}}$ amplitude on $|0\rangle$, $\frac{1}{\sqrt{2}}$ amplitude on $|1\rangle$"

OR

Is this a quantum state? ← "$\frac{i}{\sqrt{2}}$ amplitude on $|0\rangle$, 0 amplitude on $|1\rangle$"

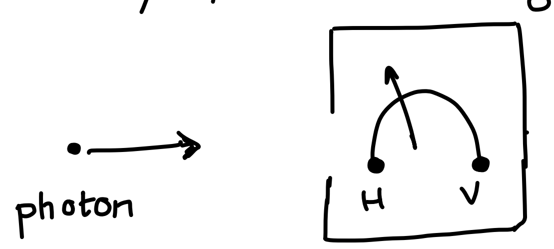
OR

"$\underbrace{\text{1 amplitude on } |0\rangle\text{, 0 amplitude on } |1\rangle}_{\text{called "}|0\rangle\text{"}}$"

called "$|0\rangle$"
reverse "$|1\rangle$"

You cannot read a quantum state, i.e., access α, β directly
Only way to extract information is via measurement

What happens if a photon in a superposition state goes into the measuring device?



α ampl. "$|0\rangle$"
β ampl. "$|1\rangle$"

This scenario is described by the second law of quantum mechanics

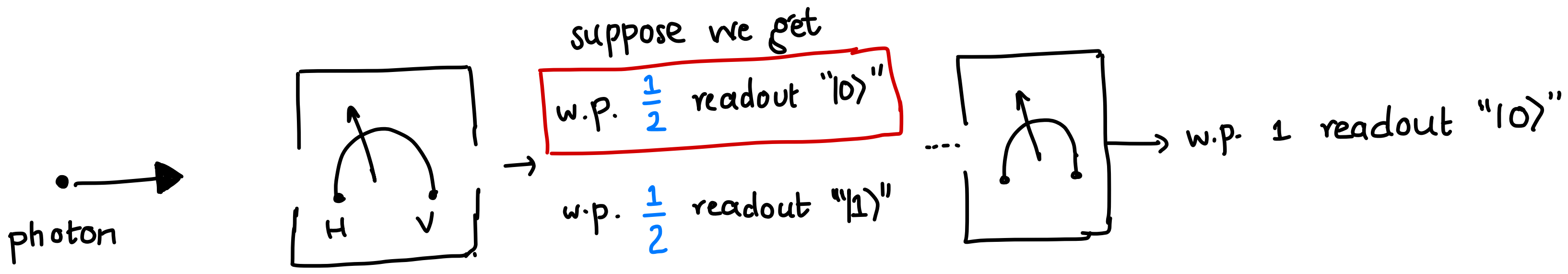
QM Law 2 For a particle with α amplitude on $|0\rangle$, β amplitude on $|1\rangle$
(Born's Rule) if you measure it, then

w/ prob $|\alpha|^2$, readout shows "$|0\rangle$" AND state becomes "1 amplitude on $|0\rangle$"
w/ prob $|\beta|^2$, readout shows "$|1\rangle$" state becomes "$\underbrace{\text{1 amplitude on } |1\rangle}_{\text{whatever outcome was observed}}$"

E.g. photon in state "$\frac{4}{5}$ amplitude on $|0\rangle$, $\frac{3}{5}$ amplitude on $|1\rangle$"

w.p. 0.64, readout shows $|0\rangle$
w.p 0.36, readout shows $|1\rangle$

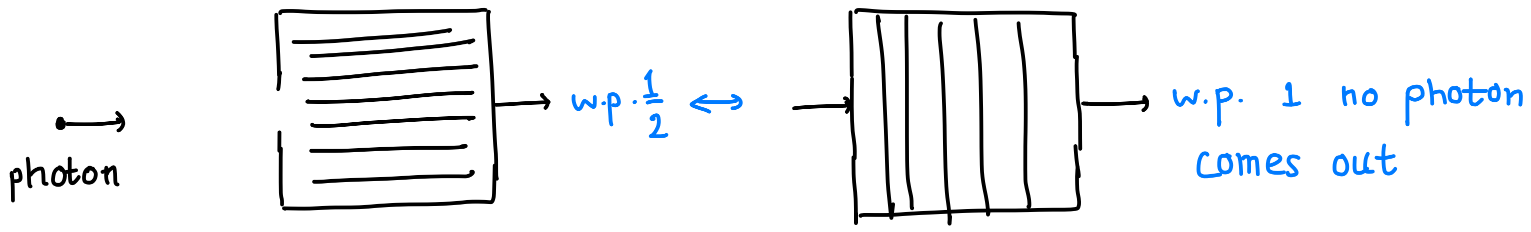
Example



$\frac{1}{\sqrt{2}}$ amplitude on $|0\rangle$

$\frac{1}{\sqrt{2}}$ amplitude on $|1\rangle$

Example (Filter)



$\frac{1}{\sqrt{2}}$ amplitude on $|0\rangle$

$\frac{1}{\sqrt{2}}$ amplitude on $|1\rangle$

Measure $\leftrightarrow$ or $\updownarrow$
Let $\leftrightarrow$ photons pass
Block $\updownarrow$ photons

To describe Born's rule more generally, we first take a detour and introduce quantum notation

Recall a qubit in state "α ampl. on "$|0\rangle$", β ampl. on "$|1\rangle$""
can be described as a column vector

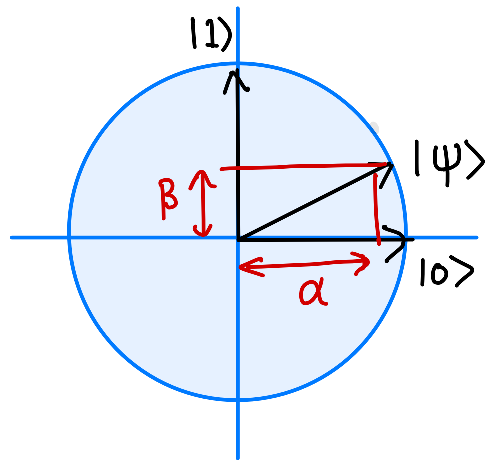
$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2 \quad \text{s.t. } |\alpha|^2 + |\beta|^2 = 1$$

It is also a unit vector

If we only use real amplitudes α, β

length of (α,β)

$= \sqrt{\alpha^2+\beta^2} = 1$



"$|0\rangle$" = 1 ampl. on "$|0\rangle$" $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
0 ampl. on "$|1\rangle$"

"$|1\rangle$" = reverse $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\{|0\rangle, |1\rangle\}$ forms an orthonormal basis for $\mathbb{R}^2$

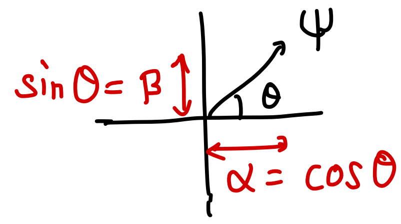
Any qubit can be written as $\alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$

Let $\alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} =$ "ψ" = unit vector

How can we describe α, β in terms of ψ ?

$\alpha =$ "Projection of "ψ" on $|0\rangle$" $= \langle$"ψ", "$|0\rangle$"$\rangle$

$\beta =$ "Projection of "ψ" on $|1\rangle$" $= \langle$"ψ", "$|1\rangle$"$\rangle$



If $u, v \in \mathbb{R}^n$, then

$$\langle u, v\rangle = \text{Inner product of } u \text{ and } v = \sum_{i=1}^{n} u_i v_i = (u_1 \cdots u_n) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u^T v$$

$$\|u\| = \text{length of } u = \sqrt{\langle u, u\rangle} = \sqrt{\sum_{i=1}^{n} u_i^2}$$

← if "ψ" $= \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

$\|\psi\| = \sqrt{\alpha^2+\beta^2}$

What happens with complex vectors $u, v \in \mathbb{C}^n$?

$$\langle u, v\rangle = \sum_{i=1}^{n} \overline{u_i} v_i = (\overline{u_1} \cdots \overline{u_n}) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = u^{\dagger} v$$

→ dagger or conjugate transpose of u

$$\|u\| = \text{length} = \sqrt{\langle u, u\rangle} = \sqrt{\sum_{i=1}^{n} \overline{u_i} u_i} = \sqrt{\Sigma |u_i|^2}$$

So, if "ψ" $= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{C}^2$, then $\|\psi\| = \sqrt{|\alpha|^2 + |\beta|^2}$

Also, $\{|0\rangle, |1\rangle\}$ is an orthonormal basis for $\mathbb{C}^2$

so, $\alpha = \langle$"ψ", "$|0\rangle$"$\rangle$ and $\beta = \langle$"ψ", "$|1\rangle$"$\rangle$

Dirac's Bra-Ket Notation

- $|\psi\rangle =$ column vector e.g. $|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$

  $|\cdot\rangle$ is called a ket

- $\langle\psi| = \underbrace{\text{conjugate transpose of } \psi}_{\psi^{\dagger} \text{ row vector}} = (\overline{\psi_1} \;\cdots\; \overline{\psi_n})$

  $\langle\cdot|$ is called a bra

- Inner Product of $|\psi\rangle, |\sigma\rangle$

$$= (\overline{\psi_1} \;\cdots\; \overline{\psi_n}) \begin{pmatrix}\sigma_1\\ \vdots \\ \sigma_n\end{pmatrix} = \langle\psi|\cdot|\sigma\rangle = \langle\psi|\sigma\rangle$$

Given a qubit $|\psi\rangle$ and orthonormal basis $\{|0\rangle, |1\rangle\}$

$$|\psi\rangle = \underbrace{\langle 0|\psi\rangle}_{\text{scalar}} |0\rangle + \underbrace{\langle 1|\psi\rangle}_{\text{scalar}} |1\rangle$$

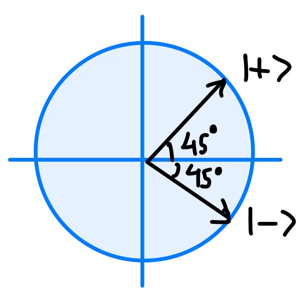
Can do the same for any basis $\{|b_0\rangle, |b_1\rangle\}$

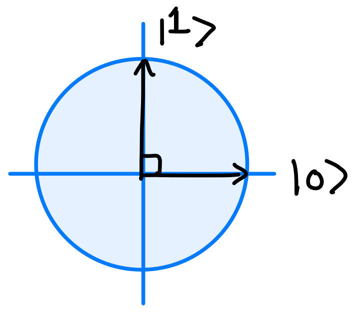
$$|\psi\rangle = \langle b_0|\psi\rangle |b_0\rangle + \langle b_1|\psi\rangle |b_1\rangle$$

Example

$$\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$= $ "$|+\rangle$"





$$\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle = \text{"}|-\rangle\text{"}$$

$$|0\rangle = \underbrace{\frac{1}{\sqrt{2}}}\, |+\rangle + \underbrace{\frac{1}{\sqrt{2}}}\, |-\rangle$$

$$|1\rangle = \underbrace{\frac{1}{\sqrt{2}}}\, |+\rangle - \underbrace{\frac{1}{\sqrt{2}}}\, |-\rangle$$

Exercise (in-class)

(1) How do you express $\langle\psi|$ in terms of ket notation?

(2) What is $|\psi\rangle\langle\psi|$?

NEXT LECTURE : Born's rule for measurement in a different basis & more ....