Probabilistic Counting and Morris Counter

Lecture 04

September 1, 2022

Part I

Counting Events

Streaming model

- The input consists of m objects/items/tokens $e_1, e_2, \ldots, e_m$ that are seen one by one by the algorithm.
- The algorithm has "limited" memory say for B tokens where $B < m$ (often $B \ll m$) and hence cannot store all the input
- Want to compute interesting functions over input

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Yes, with randomization.
"Counting large numbers of events in small registers" by Rober Morris (Bell Labs), Communications of the ACM (CACM), 1978

Probabilistic Counting Algorithm

<u>ProbabilisticCounting:</u>
$\boldsymbol{X} \leftarrow 0$
While (a new event arrives)
  Toss a biased coin that is heads with probability $1/2^{\boldsymbol{X}}$
  If (coin turns up heads)
    $\boldsymbol{X} \leftarrow \boldsymbol{X} + 1$
endWhile
Output $2^{\boldsymbol{X}} - 1$ as the estimate for the length of the stream.

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Theorem

Let $Y = 2^X$. Then $E[Y] - 1 = n$, the number of events seen.

$\log n$ vs $\log \log n$

Morris's motivation:

- Had 8 bit registers. Can count only up to $2^8 = 256$ events using deterministic counter. Had many counters for keeping track of different events and using 16 bits (2 registers) was infeasible.
- If only $\log \log n$ bits then can count to $2^{2^8} = 2^{256}$ events! In practice overhead due to error control etc. Morris reports counting up to 130,000 events using 8 bits while controlling error.

See 2 page paper for more details.

Analysis of Expectation

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Base case: $n = 0, 1$ easy to check: $X_i, Y_i - 1$ deterministically equal to $0, 1$.

Analysis of Expectation

$$
\begin{aligned}
\mathrm{E}[Y_n] &= \mathrm{E}\left[2^{X_n}\right] = \sum_{j=0}^{\infty} 2^j \Pr[X_n = j] \\
&= \sum_{j=0}^{\infty} 2^j \left(\Pr[X_{n-1} = j] \cdot (1 - \frac{1}{2^j}) + \Pr[X_{n-1} = j-1] \cdot \frac{1}{2^{j-1}} \right) \\
&= \sum_{j=0}^{\infty} 2^j \Pr[X_{n-1} = j] \\
&\quad + \sum_{j=0}^{\infty} (2 \Pr[X_{n-1} = j-1] - \Pr[X_{n-1} = j]) \\
&= \mathrm{E}[Y_{n-1}] + 1 \quad \text{(by applying induction)} \\
&= n + 1
\end{aligned}
$$

Jensen's Inequality

Definition

A real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *convex* if $f((a+b)/2) \leq (f(a)+f(b))/2$ for all a, b. Equivalently, $f(\lambda a + (1-\lambda)b) \leq \lambda f(a) + (1-\lambda) f(b)$ for all $\lambda \in [0,1]$.

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Theorem (Jensen's inequality)

Let Z be random variable with $\mathrm{E}[Z] < \infty$. If f is convex then $f(\mathrm{E}[Z]) \leq \mathrm{E}[f(Z)]$.

Implication for counter size

We have $Y_n = 2^{X_n}$. The function $f(z) = 2^z$ is convex. Hence

$$2^{\mathrm{E}[X_n]} \leq \mathrm{E}[Y_n] \leq n + 1$$

which implies

$$\mathrm{E}[X_n] \leq \log(n + 1)$$

Hence expected number of bits in counter is $\lceil \log \log(n + 1)) \rceil$.

Variance calculation

Question: Is the random variable Y_n well behaved even though expectation is right? What is its variance? Is it concentrated around expectation?

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Lemma

$\mathrm{E}\left[Y_n^2\right] = \frac{3}{2}n^2 + \frac{3}{2}n + 1$ *and hence* $Var[Y_n] = n(n-1)/2$.

Variance analysis

Analyze $\mathrm{E}\left[Y_n^2\right]$ via induction.
Base cases: $n = 0, 1$ are easy to verify since Y_n is deterministic.

$$
\begin{aligned}
E[Y_n^2] &= E[2^{2X_n}] = \sum_{j \geq 0} 2^{2j} \cdot \Pr[X_n = j] \\
&= \sum_{j \geq 0} 2^{2j} \cdot \left(\Pr[X_{n-1} = j](1 - \frac{1}{2^j}) + \Pr[X_{n-1} = j-1] \frac{1}{2^{j-1}} \right) \\
&= \sum_{j \geq 0} 2^{2j} \cdot \Pr[X_{n-1} = j] \\
&\quad + \sum_{j \geq 0} \left(-2^j \Pr[X_{n-1} = j-1] + 42^{j-1} \Pr[X_{n-1} = j-1] \right) \\
&= E[Y_{n-1}^2] + 3E[Y_{n-1}] \\
&= \frac{3}{2}(n-1)^2 + \frac{3}{2}(n-1) + 1 + 3n = \frac{3}{2}n^2 + \frac{3}{2}n + 1.
\end{aligned}
$$

Error analysis via Chebyshev inequality

We have $E[Y_n] = n$ and $Var(Y_n) = n(n-1)/2$ implies $\sigma_{Y_n} = \sqrt{n(n-1)/2} \leq n$.

Applying Cheybyshev's inequality:

$$\Pr[|Y_n - E[Y_n]| \geq tn] \leq 1/(2t^2).$$

Hence constant factor approximation with constant probability (for instance set $t = 1/2$).

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Question: Want estimate to be tighter. For any given $\epsilon > 0$ want estimate to have error at most $\epsilon \boldsymbol{n}$ with say constant probability or with probability at least $(1-\delta)$ for a given $\delta > 0$.

Part II

Improving Estimators

Probabilistic Estimation

Setting: want to compute some real-value function f of a given input I

Probabilistic estimator: a randomized algorithm that given I outputs a random answer X such that $\mathrm{E}[X] \simeq f(I)$. Estimator is *exact* if $\mathrm{E}[X] = f(I)$ for all inputs I.

Additive approximation: $|\mathrm{E}[X] - f(I)| \leq \epsilon$

Multiplicative approximation: $(1-\epsilon)f(I) \leq \mathrm{E}[X] \leq (1+\epsilon)f(I)$

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Question: Estimator only gives expectation. Bound on $Var[X]$ allows Chebyshev. Sometimes Chernoff applies. How do we improve estimator?

Variance reduction via averaging

- Run h parallel copies of algorithm with *independent* randomness
- Let $Y^{(1)}, Y^{(2)}, \ldots, Y^{(h)}$ be estimators from the h parallel copies
- Output $Z = \frac{1}{h}\sum_{i=1}^{h} Y^{(i)}$

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To run h copies need $O(\frac{1}{\epsilon^2} \log\log n)$ bits for the counters.

Error reduction via median trick

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Want:

$$\Pr[|Z_n - \mathrm{E}[Z_n]| \geq \epsilon n] \leq \delta$$

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Can set $h = \frac{1}{2\epsilon^2\delta}$ and apply Chebyshev. Better dependence on δ?

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Algorithm: Output median of $Z^{(1)}, Z^{(2)}, \ldots, Z^{(\ell)}$.

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Let Z' be median of the $\ell = c \log(1/\delta)$ independent estimators.

Lemma

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- For median estimate to be bad, more than half of A_i's have to be bad.

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- For median estimate to be bad, more than half of A_i's have to be bad.
- Using Chernoff bounds: probability of bad median is at most $2^{-c'\ell}$ for some constant c'.

Summarizing

Using variance reduction and median trick: with $O(\frac{1}{\epsilon^2}\log(1/\delta)\log\log n)$ bits one can maintain a $(1-\epsilon)$-factor estimate of the number of events with probability $(1-\delta)$. This is a *generic* scheme that we will repeatedly use.

For counter one can do (much) better by changing algorithm and better analysis. See homework and references in notes.