

Program Verification: Lecture 11

José Meseguer

Computer Science Department
University of Illinois at Urbana-Champaign

More on $\Sigma(X)$ -Algebras

Recall how we formalized the evaluation of integer arithmetic expressions with memory $m : X \rightarrow \mathbb{Z}$ as the unique $\Sigma(X)$ -homomorphism:

$$_{(\mathbb{Z}, m)} : \mathbb{T}_{\Sigma(X)} \rightarrow (\mathbb{Z}, m)$$

where (\mathbb{Z}, m) extends the integer Σ -algebra $\mathbb{Z} = (\mathbb{Z}, _{\mathbb{Z}})$ by interpreting the constants X as the memory map $m : X \rightarrow \mathbb{Z}$.

This situation is **completely general**: For **any** signature Σ and **any** Σ -algebra $\mathbb{A} = (A, _{\mathbb{A}})$, given an **assignment**, i.e., a “memory map,” $a : X \rightarrow A$, the evaluation of $\Sigma(X)$ -expressions in \mathbb{A} is the unique $\Sigma(X)$ -homomorphism:

$$_{(\mathbb{A}, a)} : \mathbb{T}_{\Sigma(X)} \rightarrow (\mathbb{A}, a)$$

Notation: $_{(\mathbb{A}, a)}$ is abbreviated to $_a : \mathbb{T}_{\Sigma(X)} \rightarrow (\mathbb{A}, a)$.

More $\Sigma(X)$ -Algebras (II)

We can summarize this situation as the following:

Fact 1: Any pair (\mathbb{A}, a) with $\mathbb{A} = (A, __{\mathbb{A}})$ a Σ -algebra and $a : X \rightarrow A$ an assignment defines a $\Sigma(X)$ -algebra (\mathbb{A}, a) .

Q: Are all $\Sigma(X)$ -algebras of this form?

A: Yes! We just need to recall the definition of an order-sorted Σ -algebra in Lecture 3:

For $\Sigma = ((S, \leq), F)$ a signature, Σ -algebra $\mathbb{A} = (A, __{\mathbb{A}})$ is just a pair with: (i) A an S -sorted set, and (ii) $__{\mathbb{A}}$ a function $__{\mathbb{A}} : f \mapsto f_{\mathbb{A}}$ interpreting each **constant** $c : \rightarrow s$ as an **element** $c_{\mathbb{A}} \in A_s$, and each **symbol** $f : w \rightarrow s$ in Σ as a **function** $f_{\mathbb{A}} \in [A^w \rightarrow A_s]$, with f overloaded agreeing on common data.

More $\Sigma(X)$ -Algebras (III)

Therefore, if $\Sigma = ((S, \leq), F)$, then $\Sigma(X) = ((S, \leq), F \uplus X)$, where each new constant $x \in X_s$ has typing $x : \rightarrow s$, and \uplus denotes **disjoint union** of sets.

Recall from STACS that if sets U and V are disjoint, any function $h : U \uplus V \rightarrow W$ decomposes **uniquely** as a pair $(h|_U : U \rightarrow W, h|_V : V \rightarrow W)$ of its **restrictions** to U and V .

Therefore, if $\mathbb{B} = (B, \neg_{\mathbb{B}})$ is a $\Sigma(X)$ -algebra, then $\neg_{\mathbb{B}}$ decomposes **uniquely** as a pair $(\neg_{\mathbb{B}}|_F, \neg_{\mathbb{B}}|_X)$. But note that $\neg_{\mathbb{B}}|_X : X \rightarrow B$ is just an **assignment**! and $(B, \neg_{\mathbb{B}}|_F)$ is just a Σ -algebra! **Notation:** $(B, \neg_{\mathbb{B}}|_F) = \mathbb{B}|_{\Sigma}$, the Σ -**reduct** of \mathbb{B} .

Fact 2: $\mathbb{B} = (B, \neg_{\mathbb{B}})$ decomposes **uniquely** as $\mathbb{B} = (\mathbb{B}|_{\Sigma}, \neg_{\mathbb{B}}|_X)$.

More $\Sigma(X)$ -Homomorphisms

Facts 1 and 2 tell us that any $\Sigma(X)$ -algebra is **exactly the same thing** as a pair (\mathbb{A}, a) with \mathbb{A} a Σ -algebra and $a \in [X \rightarrow A]$ an assignment.

Q: What is a $\Sigma(X)$ -**homomorphism** $h : (\mathbb{A}, a) \rightarrow (\mathbb{C}, c)$?

A: The answer is summarized in **Fact 3** below.

Fact 3: Since h must preserve **both** the Σ -symbols F and the constants X but $F \cap X = \emptyset$, h is exactly:

1. a Σ -homomorphism $h : \mathbb{A} \rightarrow \mathbb{C}$ such that
2. for each $s \in S$, $x \in X_s$, $h_s(a(x)) = c(x)$, i.e., $a; h = c$.

Example: Substitutions Revisited

Let us apply **Fact 2** to the **initial** $\Sigma(X)$ -**algebra** $\mathbb{T}_{\Sigma(X)} = (T_{\Sigma(X)}, -\mathbb{T}_{\Sigma(X)})$. What **unique decomposition** do we get for $\mathbb{T}_{\Sigma(X)}$? We get a pair $(\mathbb{T}_{\Sigma(X)}|_{\Sigma}, \eta_X)$, where:

1. $\mathbb{T}_{\Sigma(X)}|_{\Sigma} = (T_{\Sigma(X)}, -\mathbb{T}_{\Sigma(X)}|_F)$, that is, the elements $t \in T_{\Sigma(X)}$ are the **same**: (Σ -terms with variables in X), but **only** the Σ -operations are considered; and
2. $\eta_X : X \rightarrow T_{\Sigma(X)} : x \mapsto x$ is the **identity interpretation** for each variable x in X , that is, the **identity substitution**.

To simplify the notation, we will denote $\mathbb{T}_{\Sigma(X)}|_{\Sigma}$ by $\mathbb{T}_{\Sigma}(X)$, and will call it the **free Σ -algebra on the variables X** .

Example: Substitutions Revisited (II)

Consider now another Σ -sorted set Y of variables and a **substitution** $\theta : X \rightarrow T_{\Sigma(Y)}$.

Q: how can we **model** the extension of θ to the map on terms $_ \theta : T_{\Sigma(X)} \rightarrow T_{\Sigma(Y)}$ defined in Lecture 3?

A: Easy! Consider the $\Sigma(X)$ -algebra $(\mathbb{T}_{\Sigma(Y)}, \theta)$. Then $_ \theta$ is just the **unique** $\Sigma(X)$ -homomorphism:

$$_ \theta : \mathbb{T}_{\Sigma(X)} \rightarrow (\mathbb{T}_{\Sigma(Y)}, \theta),$$

which decomposing $\mathbb{T}_{\Sigma(X)}$ as $\mathbb{T}_{\Sigma(X)} = (\mathbb{T}_{\Sigma(X)}, \eta_X)$, is the unique $\Sigma(X)$ -homomorphism:

$$_ \theta : (\mathbb{T}_{\Sigma(X)}, \eta_X) \rightarrow (\mathbb{T}_{\Sigma(Y)}, \theta).$$

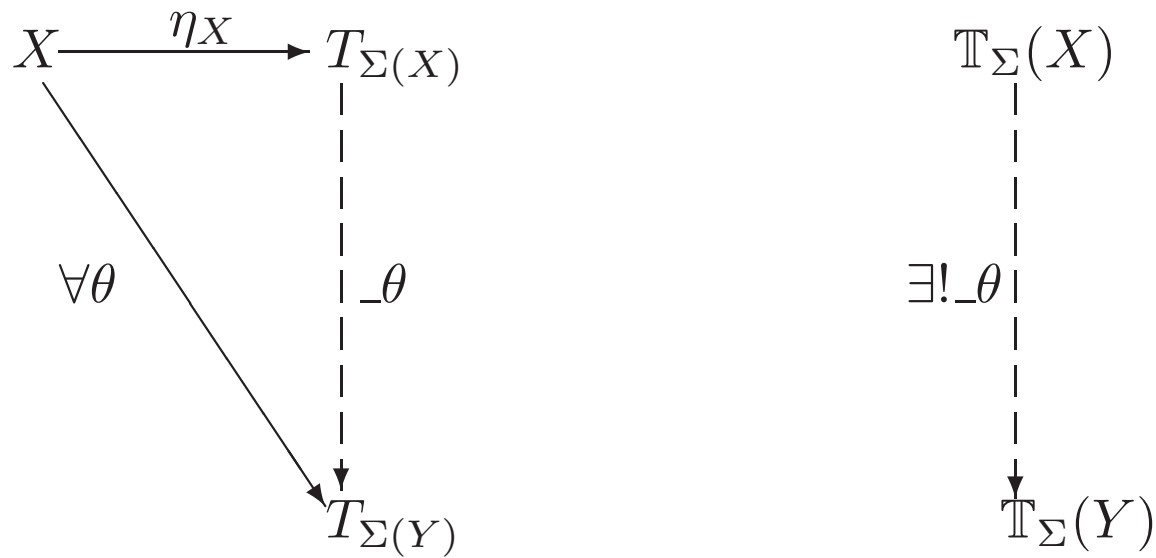
Example: Substitutions Revisited (III)

But by **Fact 3**, $_ \theta : (\mathbb{T}_\Sigma(X), \eta_X) \rightarrow (\mathbb{T}_\Sigma(Y), \theta)$ is a $\Sigma(X)$ -homomorphism iff:

1. $_ \theta : \mathbb{T}_\Sigma(X) \rightarrow \mathbb{T}_\Sigma(Y)$ is a Σ -homomorphism, and
2. $\eta_X; _ \theta = \theta$

Therefore, each substitution θ has a **unique extension** to a Σ -homomorphism $_ \theta$ such that the following diagram commutes:

Homomorphic Extension of Substitutions



\mathbf{Set}^S : S-Indexed Families and S-Indexed Functions \mathbf{Alg}_{Σ} : Σ -Algebras and Σ -Homomorphism

Freeness Theorem

The extension $\theta \mapsto _ \theta$ is an instance of the more general:

Theorem (Freeness Theorem). For each Σ -algebra $\mathbb{A} = (A, _ \mathbb{A})$, and assignment $a : X \longrightarrow A$ there exists a **unique** Σ -homomorphism $_ a : \mathbb{T}_\Sigma(X) \longrightarrow \mathbb{A}$ such that $\eta_X; _ a = a$.

Proof: Since (\mathbb{A}, a) is a $\Sigma(X)$ -algebra, by the initiality of $\mathbb{T}_{\Sigma(X)}$ there is a **unique** $\Sigma(X)$ -homomorphism

$$_ a : \mathbb{T}_{\Sigma(X)} \rightarrow (\mathbb{A}, a),$$

which decomposing $\mathbb{T}_{\Sigma(X)}$ as $\mathbb{T}_{\Sigma(X)} = (\mathbb{T}_\Sigma(X), \eta_X)$, is the same thing as a **unique** $\Sigma(X)$ -homomorphism:

$$_ a : (\mathbb{T}_\Sigma(X), \eta_X) \rightarrow (\mathbb{A}, a),$$

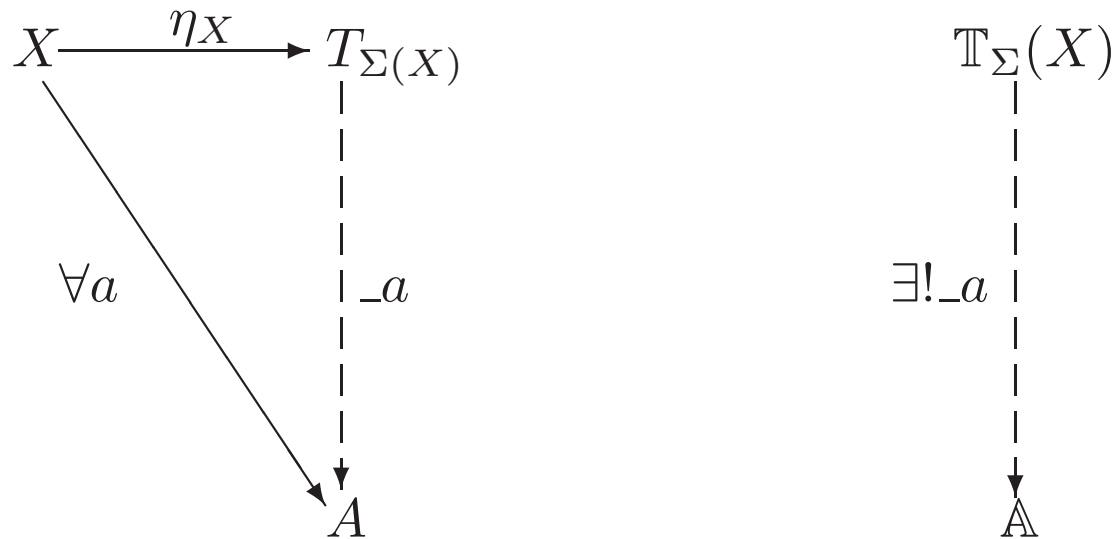
which by the definition of $\Sigma(X)$ -homomorphism is the same thing as a **unique** Σ -homomorphism

$$_a : \mathbb{T}_\Sigma(X) \rightarrow \mathbb{A}$$

such that $\eta_X; _a = a$, as desired. q.e.d.

This theorem can be summarized in the following diagram:

$T_\Sigma(X)$ as a Free Σ -Algebra on X



\mathbf{Set}^S : S-Indexed Families and S-Indexed Functions \mathbf{Alg}_Σ : Σ -Algebras and Σ -Homomorphism

Useful Corollary on Free Σ -Algebras

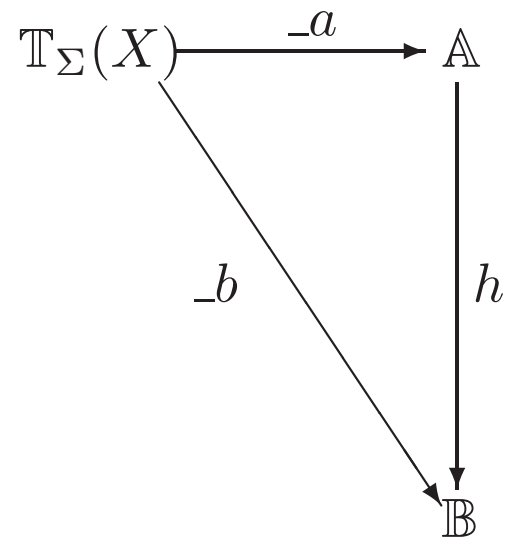
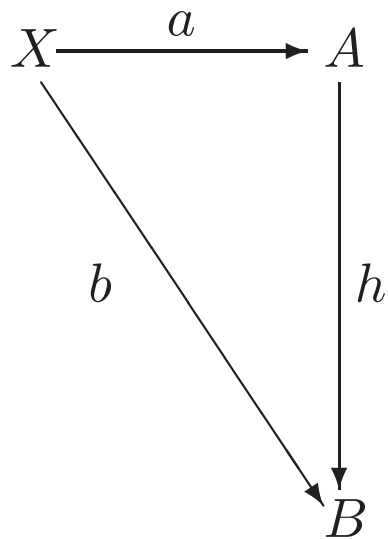
Corollary (Freeness Corollary). For any Σ -homomorphism $h : \mathbb{A} \longrightarrow \mathbb{B}$, and assignments $a : X \longrightarrow A$, $b : X \longrightarrow B$ such that $a;h = b$, the following identity between Σ -homomorphisms holds:

$$_a;h = _b$$

Proof: $_a;h$ is a Σ -homomorphism $_a;h : \mathbb{T}_\Sigma(X) \longrightarrow \mathbb{B}$. But since, by hypothesis, we have $a;h = b$, we must also have: $\eta_X; _a;h = a;h = b$, which by the Freeness Theorem forces $_a;h = _b$, as desired. q.e.d.

The corollary can be summarized in the following diagram:

Useful Corollary on Free Σ -Algebras (II)



\mathbf{Set}^S : S-Indexed Families and S-Indexed Functions \mathbf{Alg}_Σ : Σ -Algebras and Σ -Homomorphism

What is “free” about a Free Algebra?

Clearly, the concept of a free Σ -algebra $\mathbb{T}_\Sigma(X)$ **generalizes** the case of an initial algebra, since when $X = \emptyset$, where \emptyset here denotes the S -indexed set having all its components empty, we have $\mathbb{T}_\Sigma(\emptyset) = \mathbb{T}_\Sigma$. As in the case of initial algebras, free algebras have (provided Σ is sensible) **no confusion**. Therefore, the first meaning of “free” is that **no equalities force terms** in $\mathbb{T}_\Sigma(X)$ to **become equal**: they are all different, unconstrained, and in this sense “free.”

Note that if X is nonempty $\mathbb{T}_\Sigma(X)$ has **junk**! (even though, $\mathbb{T}_{\Sigma(X)}$, with the same data elements, doesn’t!). Which junk? Well, X , of course, and all the junk spread by X when building terms with variables. However, this “junk” is very well-behaved.

What is “free” about a Free Algebra? (II)

X is well-behaved: we can **freely interpret** the variables in X as data elements in any Σ -algebra \mathbb{B} by **any** assignment $b : X \longrightarrow B$ with the guarantee that b will always **extend** to a **unique** Σ -homomorphism $_b$. This **free interpretation** and **free extensibility** is the second meaning of “free.”

This freedom is not enjoyed by other algebras. Let Σ be the unsorted signature with constant 0 and unary s . \mathbb{T}_Σ is the natural numbers in Peano notation. Define $\mathbb{T}_\Sigma \cup \{x, y, z\}$ with elements $T_\Sigma \cup \{x, y, z\}$, with 0 and s interpreted as before on the T_Σ part, and with $s(x) = y$, $s(y) = z$, and $s(z) = x$. Now the junk $X = \{x, y, z\}$ is badly behaved. Let \mathbb{N} be the natural numbers in decimal notation with 0 and successor. There is **no assignment at all** $b : X \longrightarrow \mathbb{N}$ that can be extended to a Σ -homomorphism $\mathbb{T}_\Sigma \cup \{x, y, z\} \longrightarrow \mathbb{N}$.

Satisfaction of Equations

Let $X = \{X_s\}$ be such that for each $s \in S$, X_s is a countably infinite set. Given a Σ -algebra \mathbb{A} , an assignment $a : X \longrightarrow A$, and a Σ -equation $t = t'$ with variables in X , we define the **satisfaction relation** $(\mathbb{A}, a) \models t = t'$ by means of the equivalence,

$$(\mathbb{A}, a) \models t = t' \quad \Leftrightarrow \quad t \, a = t' \, a.$$

We then define the **satisfaction relation** $\mathbb{A} \models t = t'$ iff for **all** assignments $a : X \longrightarrow A$ we have $(\mathbb{A}, a) \models t = t'$.

Note that, since each (\mathbb{A}, a) is a $\Sigma(X)$ -algebra, we have defined the satisfaction of $\mathbb{A} \models t = t'$ as the satisfaction of the **ground** $\Sigma(X)$ -equation $t = t'$ by each (\mathbb{A}, a) , denoted $(\mathbb{A}, a) \models t = t'$, for **all** assignments a .

Examples of Satisfaction

Consider the unsorted signature Σ with constants $0, 1$, and operations of addition $_ + _$, and multiplication $_ * _$. Then all the algebras \mathbb{N} , \mathbb{N}_k , \mathbb{Z} , \mathbb{Q} , \mathbb{R} , \mathbb{C} , in Lecture 3, pgs. 3–5, satisfy the equations:

- $x + 0 = x$
- $x + y = y + x$
- $x + (y + z) = (x + y) + z$
- $x * 1 = x$
- $x * y = y * x$
- $x * (y * z) = (x * y) * z$

Examples of Satisfaction (II)

Consider the signature Σ for Boolean operations in page 6 of Lecture 3. Then the Σ -algebras \mathbb{B} and $\mathbb{P}(X)$ satisfy the equations:

- $x \text{ and } \text{true} = x \quad (\forall x) \quad x \text{ or } \text{false} = x$
- $x \text{ and } y = y \text{ and } x \quad (\forall x, y) \quad x \text{ or } y = y \text{ or } x$
- $x \text{ and } (y \text{ and } z) = (x \text{ and } y) \text{ and } z$
- $x \text{ or } (y \text{ or } z) = (x \text{ or } y) \text{ or } z$
- $x \text{ and } x = x \quad x \text{ or } x = x$

Examples of Satisfaction (III)

Consider the NAT-LIST signature in Lecture 2, and the two algebras for it defined in Lecture 4, pages 4–5. Show that the first algebra (where the sort `List` is interpreted as finite strings of natural numbers) satisfies all the equation in the module NAT-LIST.

Show also that the second algebra (where the sort `List` is interpreted as finite sets of natural numbers) does **not** satisfy the equation

$$\text{eq } \text{length}(N \text{ . } L) = s \text{ length}(L) \text{ .}$$

Examples of Satisfaction (IV)

Consider all the examples 1–3 of algebras for the “vector-space-like” signature of Picture 4.1 defined in pages 5–6 of Lecture 4. Prove that, for x, y variables of sort **Scalar**, and v, v' variables of sort **Vector**, all these algebras satisfy the equations:

- $(x + y).v = (x.v) + (y.v)$
- $x.(v + v') = (x.v) + (x.v')$
- $0.v = \vec{0}$
- $1.v = v$

Examples of Satisfaction (V)

A **permutation** on n elements is a bijective function $\pi : [n] \longrightarrow [n]$, where $[n] = \{1, \dots, n\}$. The set of all such permutations is denoted S_n and has function composition as a binary operation $;$ for which the identity permutation $1_{[n]} : [n] \longrightarrow [n]$ is an identity element. Also, for each $\pi \in S_n$ the inverse function π^{-1} is another permutation such that, $\pi; \pi^{-1} = 1_{[n]} = \pi^{-1}; \pi$. S_n is called the **symmetric group** on n elements, because it satisfies the **group theory** axioms,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad (\text{associativity})$$

$$x \cdot 1 = x = 1 \cdot x \quad (\text{identity})$$

$$x \cdot x^{-1} = 1 = x^{-1} \cdot x \quad (\text{inverse})$$

Similarly, given a set X of elements, the set X^* of its strings with the concatenation operation is a **monoid**, because it satisfies the above associativity and identity axioms.

Models and Theorems of Theories

Given an order-sorted equational theory (Σ, E) and a Σ -algebra \mathbb{A} , we write $\mathbb{A} \models (\Sigma, E)$, or, equivalently, $\mathbb{A} \models E$, iff \mathbb{A} satisfies all the equations in E . We then call \mathbb{A} a **model** of (Σ, E) , or a (Σ, E) -**algebra**. For example, for (Σ, E) the theory groups (resp. monoids), a model of (Σ, E) is called a group (resp. a monoid).

Given a theory (Σ, E) , what other equations, besides those in E , does any (Σ, E) -algebra satisfy? We call an equation $t = t'$ a **theorem** of (Σ, E) iff for each (Σ, E) -algebra \mathbb{A} we have, $\mathbb{A} \models t = t'$. We then write $(\Sigma, E) \models t = t'$.

We have now two different relations: (i) $(\Sigma, E) \vdash t = t'$, telling us which equations we can mechanically **prove**, and (ii) $(\Sigma, E) \models t = t'$, telling us which equations are **theorems**.

Soundness and Completeness

There are now two obvious questions:

Soundness: Does the implication

$$(\Sigma, E) \vdash t = t' \Rightarrow (\Sigma, E) \models t = t$$

always hold? That is, is anything we can **prove** always **true**, i.e., always **a theorem**? For example, we can prove the equations $1^{-1} = 1$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$ from the theory of groups, but are they really theorems of group theory?

Completeness: Does the implication

$$(\Sigma, E) \models t = t' \Rightarrow (\Sigma, E) \vdash t = t$$

always hold? That is, can we **prove** all the equations that are **theorems** of (Σ, E) ?

Soundness Theorem

Soundness Theorem. For (Σ, E) an equational theory with Σ sensible, kind-complete, and with nonempty sorts, and for all Σ -equations $t = t'$, we have the implication:

$$(\Sigma, E) \vdash t = t' \quad \Rightarrow \quad (\Sigma, E) \models t = t'.$$

Proof: Note that, by definition, we have

$$(\Sigma, E) \vdash t = t' \Leftrightarrow t =_E t' \Leftrightarrow (\Sigma, \overrightarrow{E} \cup \overleftarrow{E}) \vdash t \rightarrow^* t'.$$

Therefore, what we have to prove is the implication

$$(\Sigma, \overrightarrow{E} \cup \overleftarrow{E}) \vdash t \rightarrow^* t' \quad \Rightarrow \quad (\Sigma, E) \models t = t'.$$

We can do so by induction on the **length** of the rewrite sequence $t \rightarrow^* t'$.

Soundness Theorem (II)

Base Case. If the length of $t \rightarrow^* t'$ is 0, then t' is **identical** to t , so we need to prove $(\Sigma, E) \models t = t$, which trivially holds, since for **any** Σ -algebra \mathbb{A} we have $\mathbb{A} \models t = t$. In particular, if $\mathbb{A} \models E$, then, of course, $\mathbb{A} \models t = t$.

Induction Step. Assume that if $(\Sigma, \overrightarrow{E} \cup \overleftarrow{E}) \vdash t \rightarrow^* w$ and the sequence $t \rightarrow^* w$ has length n , then the relation $(\Sigma, E) \models t = w$ holds, and consider an additional rewrite step $w \rightarrow_{\overrightarrow{E} \cup \overleftarrow{E}} t'$. We then need to prove that $(\Sigma, E) \models t = t'$. We will be done if we can prove:

Lemma. For all w, t' , if $w \rightarrow_{\overrightarrow{E} \cup \overleftarrow{E}} t'$ then $(\Sigma, E) \models w = t'$.

Soundness Theorem (III)

Indeed, if this Lemma holds, then for each Σ -algebra \mathbb{A} such that $\mathbb{A} \models E$ and each assignment a we have $(\mathbb{A}, a) \models t = w$ (by Ind. Hyp.), and $(\mathbb{A}, a) \models w = t'$ (by Lemma). That is,

$$t a = w a \quad \wedge \quad w a = t'' a$$

and therefore $(\mathbb{A}, a) \models t = t''$, so that $(\Sigma, E) \models t = t'$.

Proof of the Lemma. We must prove the implication $w \rightarrow_{\vec{E} \cup \overleftarrow{E}} t' \Rightarrow (\Sigma, E) \models w = t'$. But the rewrite $w \rightarrow_{\vec{E} \cup \overleftarrow{E}} t'$ uses an equation $(u = v) \in E$ either from left to right or from right to left at some position p in w and with some substitution $\theta : X \rightarrow T_{\Sigma(X)}$, so that, if $u = v$ is applied left-to-right, $w = w[u\theta]_p$ and $t' = w[v\theta]_p$.

We prove the case where $u = v$ is applied from left to right. The right-to-left case is completely similar.

Soundness Theorem (IV)

The proof is by induction of the length $|p|$ of the position p .

Base Case. If $|p| = 0$, then $p = \epsilon$ is the empty string.

Therefore we have $w = u\theta$ and $t' = v\theta$, and we need to prove that for each \mathbb{A} such that $\mathbb{A} \models E$ and each assignment a we have $(\mathbb{A}, a) \models u\theta = v\theta$, that is, that $u\theta a = v\theta a$.

But, since $_ \theta; _ a$ is a Σ -homomorphism and $\eta_X; _ \theta; _ a = \theta; _ a$, by the Freeness Theorem we have:

$$_ \theta; _ a = _ (\theta; _ a)$$

And since $\mathbb{A} \models E$ and $(\theta; _ a) \in [X \rightarrow A]$, in particular, $(\mathbb{A}, (\theta; _ a)) \models u = v$, that is, $u\theta a = v\theta a$, as desired.

Soundness Theorem (V)

Induction Hypothesis. We assume that the Lemma holds for $|p| = n$. Consider now $w = w[u\theta]_{i.p}$ and $t' = w[v\theta]_{i.p}$, with $|i.p| = n + 1$. This means that, for some f , $w = f(w_1, \dots, w_n)$, $1 \leq i \leq n$, $w = f(w_1, \dots, w_i[u\theta]_p, \dots, w_n)$ and $t' = f(w_1, \dots, w_i[v\theta]_p, \dots, w_n)$.

But by the Ind. Hyp., if $\mathbb{A} \models E$ then $\mathbb{A} \models w_i[u\theta]_p = w_i[v\theta]_p$. Therefore, for any assignment $a \in [X \rightarrow A]$ we have:

$$w a = f_{\mathbb{A}}(w_1 a, \dots, w_i[u\theta]_p a, \dots, w_n a) = f_{\mathbb{A}}(w_1 a, \dots, w_i[v\theta]_p a, \dots, w_n a) = t' a$$

as desired. q.e.d.

This also concludes the proof of the Theorem. q.e.d.

Exercises

Ex.10.1 For a subsignature $\Sigma = ((S, \leq), F) \subseteq \Sigma' = ((S, \leq), F')$ and $\mathbb{A} = (A, __{\mathbb{A}})$ a Σ' -algebra, define its Σ -**reduct** $\mathbb{A}|_{\Sigma}$ as the Σ -algebra $\mathbb{A}|_{\Sigma} = (A, __{\mathbb{A}}|_F)$. Prove that for any Σ -equation $u = v$ we have the equivalence:

$$\mathbb{A} \models u = v \quad \Leftrightarrow \quad \mathbb{A}|_{\Sigma} \models u = v.$$

Ex.10.2 (i) Let $h : \mathbb{A} \longrightarrow \mathbb{B}$ be a Σ -isomorphism, and $u = v$ a Σ -equation. Prove that

$$\mathbb{B} \models u = v \quad \Leftrightarrow \quad \mathbb{A} \models u = v.$$

(ii) Give an example of a bijective Σ -homomorphism h such that the above equivalence does not hold (**Hint**: Consider order-sorted signatures Σ that are not kind-complete).

Exercises (II)

Ex.10.3 Call a Σ -algebra \mathbb{A} a **subalgebra** of a Σ -algebra \mathbb{B} iff for each sort $s \in S$ we have $A_s \subseteq B_s$, and the S -family of inclusion functions $j = \{j_s : A_s \longrightarrow B_s\}_{s \in S}$, with $j_s : a \mapsto a$ mapping each element $a \in A_s$ identically to itself is a Σ -homomorphism $j : \mathbb{A} \longrightarrow \mathbb{B}$. We then write: $\mathbb{A} \subseteq \mathbb{B}$. Show that if $\mathbb{A} \subseteq \mathbb{B}$, for any Σ -equation $u = v$ we have:

$$\mathbb{B} \models u = v \quad \Rightarrow \quad \mathbb{A} \models u = v$$

Give an example showing that the implication in the other direction in general does not hold.

Exercises (II)

Ex.10.4 Let $h : \mathbb{A} \longrightarrow \mathbb{B}$ be a surjective Σ -homomorphism, and $u = v$ a Σ -equation. Prove that

$$\mathbb{A} \models u = v \quad \Rightarrow \quad \mathbb{B} \models u = v$$

Show, by giving a counterexample, that the implication in the other direction in general does not hold.