Appendix to Lecture 9: Sufficient Completeness Theorem (for $B = \emptyset$)

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Theorem. Let (Σ, E) be an equational theory such that for each equation $u = v$ in E $vars(v) \subseteq vars(u)$ holds, and the rules $\vec{E}$ are terminating. Then $(\Sigma, \vec{E})$ is sufficiently complete with respect to a constructor subsignature $\Omega \subseteq \Sigma$ iff $D \setminus (Red \cup Ctor) = \emptyset$, where:

- $Ctor = T_\Omega$
- $Red = \{t \in T_\Sigma \mid t \neq t!_{\vec{E}}\}$
- $D = \{f(u_1, \ldots, u_n) \in T_\Sigma \mid n \geq 0 \ \wedge \ u_i \in T_\Omega, \ 1 \leq i \leq n, \ \wedge f \in \Sigma \setminus \Omega\}$.

Proof: The ($\Rightarrow$) implication is proved by contradiction. Suppose that $(\Sigma, \vec{E})$ is sufficiently complete but $f(u_1, \ldots, u_n) \in D \setminus (Red \cup Ctor)$, $n \geq 0$. Then, by construction, $f(u_1, \ldots, u_n) \notin T_\Omega$, and $f(u_1, \ldots, u_n) = f(u_1, \ldots, u_n)!_{\vec{E}}$, contradicting the sufficient completeness assumption that $f(u_1, \ldots, u_n)!_{\vec{E}} \in T_\Omega$.

The ($\Leftarrow$) implication is also proved by contradiction. Suppose that $D \setminus (Red \cup Ctor) = \emptyset$ but $(\Sigma, \vec{E})$ is not sufficiently complete. Then there is a term $t \in T_\Sigma$ such that $t!_{\vec{E}} \notin T_\Omega$. But then there exists a subterm $u \trianglelefteq t!_{\vec{E}}$ such that $u \notin T_\Omega$ and u is a smallest possible subterm with that property in the $\trianglelefteq$ order. Of course, $u = u!_{\vec{E}}$. Then either, (i) $u = a$, with a constant $a \in \Sigma \setminus \Omega$, or (ii) u is a term of the form $u = f(u_1, \ldots, u_n)$, where, by $\trianglelefteq$-minimality, $u_i \in T_\Omega$, $1 \leq i \leq n$, and, by $u \notin T_\Omega$, $f \in \Sigma \setminus \Omega$. Therefore, in cases either (i) or (ii), $u \in D$. But since $u \notin T_\Omega$ and $u = u!_{\vec{E}}$, $u \in D \setminus (Red \cup Ctor)$, contradicting $D \setminus (Red \cup Ctor) = \emptyset$. $\square$

When the constructors Ω are free, a smaller subset *Red* of reducible terms can be chosen, as shown by the following corollary.

Corollary. Let (Σ, E) be an equational theory such that for each equation $u = v$ in E $vars(v) \subseteq vars(u)$ holds, and the rules $\vec{E}$ are terminating. Assume, furthermore, that the constructors Ω are *free*,[1] that is, for each $u \in T_\Omega$, $u = u!_{\vec{E}}$. Then $(\Sigma, \vec{E})$ is sufficiently complete with respect to a constructor subsignature $\Omega \subseteq \Sigma$ iff $D \setminus (Red \cup Ctor) = \emptyset$, where:

- $Ctor = T_\Omega$
- $Red = \{u\theta \in T_\Sigma \mid (u = v) \in E \ \wedge \ \theta \in [vars(u) \to T_\Omega]\}$
- $D = \{f(u_1, \ldots, u_n) \in T_\Sigma \mid n \geq 0 \ \wedge \ u_i \in T_\Omega, \ 1 \leq i \leq n, \ \wedge f \in \Sigma \setminus \Omega\}$.

Proof: The ($\Rightarrow$) implication is proved by contradiction. Suppose that $(\Sigma, \vec{E})$ is sufficiently complete but $f(u_1, \ldots, u_n) \in D \setminus (Red \cup Ctor)$, $n \geq 0$. Then, by construction, $f(u_1, \ldots, u_n) \notin T_\Omega$, and $u_i \in T_\Omega$, $1 \leq i \leq n$. By the free constructor assumption we also have $u_i = u_i!_{\vec{E}}$, $1 \leq i \leq n$. Furthermore, $f(u_1, \ldots, u_n) = f(u_1, \ldots, u_n)!_{\vec{E}}$, because, otherwise, we should have a rewrite $f(u_1, \ldots, u_n) \to w$ at the top position ε, forcing $f(u_1, \ldots, u_n) \in Red$, wich is impossible, since $f(u_1, \ldots, u_n) \in D \setminus (Red \cup Ctor)$. But $f(u_1, \ldots, u_n) = f(u_1, \ldots, u_n)!_{\vec{E}}$ and $f(u_1, \ldots, u_n) \notin T_\Omega$ contradict the sufficient completeness assumption that $f(u_1, \ldots, u_n)!_{\vec{E}} \in T_\Omega$.

The ($\Leftarrow$) implication is also proved by contradiction. Suppose that $D \setminus (Red \cup Ctor) = \emptyset$ but $(\Sigma, \vec{E})$ is not sufficiently complete. Then there is a term $t \in T_\Sigma$ such that $t!_{\vec{E}} \notin T_\Omega$. But then there exists a subterm $u \trianglelefteq t!_{\vec{E}}$ such that $u \notin T_\Omega$ and u is a smallest possible subterm with that

[1] Constructor freedom can be guaranteed by checking that for each $u = v$ in E and for each variable specialization ρ of $vars(u)$, $u\rho \notin T_{\Omega(X)}$.

property in the $\trianglelefteq$ order. Of course, $u = u!_{\vec{E}}$. Then either, (i) $u = a$, with a constant $a \in \Sigma \setminus \Omega$, or (ii) u is a term of the form $u = f(u_1, \ldots, u_n)$, where, by $\trianglelefteq$-minimality, $u_i \in T_\Omega$, $1 \leq i \leq n$, and, by $u \notin T_\Omega$, $f \in \Sigma \setminus \Omega$. Therefore, in cases either (i) or (ii), $u \in D$. And since $u \notin T_\Omega$, $u \in D \setminus \mathit{Ctor}$. Furthermore, since $u = u!_{\vec{E}}$ and constructors are free, reasoning as in the proof of ($\Rightarrow$) we must also have $u \in D \setminus \mathit{Red}$, and therefore $u \in D \setminus (\mathit{Red} \cup \mathit{Ctor})$, contradicting $D \setminus (\mathit{Red} \cup \mathit{Ctor}) = \emptyset$. $\square$