Program Verification: Lecture 6

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Executability Conditions

A functional module `fmod` (Σ, E) `endfm` with constructor subsignature $\Omega \subseteq \Sigma$ satisfying: (1) **Unique termination**, (2) **Sufficient Completenes** and (3) **Sort Preservation**, and, also, $(\forall t \in T_\Omega)\ t!_{\vec{E}} = t$, has a canonical term algebra $\mathbb{C}_{\Sigma/E}$ as its semantics.

Conditions (1)–(3), plus requirement $(\forall t \in T_\Omega)\ t!_{\vec{E}} = t$, can best be understood by noting that the `red` command mapping $t \in T_\Sigma$ to $t!_{\vec{E}} \in T_\Omega$ is just rewriting t to termination with the rules $\vec{E}$.

But a functional module can have axioms B. That is, it can be of the form `fmod` $(\Sigma, E \cup B)$ `endfm`. Then, the `red` command simplifies $t \in T_\Sigma$ to $t!_{\vec{E}/B} \in T_\Omega$ with the rules $\vec{E}$ modulo B.

What executability conditions ensure that $\mathbb{C}_{\Sigma/E,B}$ exists for `fmod` $(\Sigma, E \cup B)$ `endfm` in general? They will be executability requirements on the rewrite theory $(\Sigma, B, \vec{E})$.

Executability Conditions (II)

In terms of the rewrite theory $(\Sigma, B, \vec{E})$,

1. **Unique Termination** will follow from $\vec{E}$ being:
    - terminating modulo B, and
    - confluent modulo B
2. **Sufficiently Completeness** will follow from $\vec{E}$ being so modulo B, and
3. **Sort Preservation** will follow from $\vec{E}$ being sort decreasing.

The requirement $(\forall t \in T_\Omega)\ t!_{\vec{E}/B} = t$ will not be needed: it was just a simplifying assumption. And we will make explicit an implicit assumption on variables in the rules $\vec{E}$ essential for executability.

Under the above executability requirements we will then define the canonical term algebra $\mathbb{C}_{\Sigma/E,B}$ of a functional module `fmod` $(\Sigma, E \cup B)$ `endfm` with constructors $\Omega \subseteq \Sigma$ in full generality.

No Extra Variables in Righthand Sides

Consider the rule $0 \to x * 0$. This rule is problematic: we have to guess how to instantiate the variable x in $x * 0$ before applying it, and there is an infinite number of instantiations for x.

Instead, the rule $x * 0 \to 0$ can be applied without problems, since the same substitution obtained by matching for the lefthand side can be reused to generate the righhand side replacement.

Therefore, for any functional module `fmod` $(\Sigma, E \cup B)$ `endfm` and associated rewrite theory $(\Sigma, B, \vec{E})$ we will require:

> *For each $t \to t' \in \vec{E}$, any variable x occuring in t' must also occur in t, i.e., $vars(t') \subseteq vars(t)$.*

Sort Decreasingness

Another important requirement on $(\Sigma, B, \vec{E})$ is:

> *(SD) Sort-decreasingness: For each $t \to t' \in \vec{E}$, $s \in S$, and each substitution θ we have $t\theta : s \;\Rightarrow\; t'\theta : s$.*

where $t : s$ abbreviates $t \in T_{\Sigma,s}$. Prove by well-founded induction on the context C below which a rewrite $C[t\theta] \to_{\vec{E}} C[t'\theta]$ takes place, that under condition (SD), if $u \to_{\vec{E}} v$, then $u : s \;\Rightarrow\; v : s$.

To see why without sort-decreasingness things can go wrong, let Σ have sorts C and D with $C < D$, a constant c of sort C, a constant d of sort D, and a subsort-overloaded unary function $f : C \longrightarrow C$, $f : D \longrightarrow D$. Let $B = \varnothing$ and $\vec{E} = \{c \to d, f(f(x : C)) \to f(x : C)\}$. With the second rule $f(f(c))$ rewrites to $f(c)$, and then to $f(d)$ with the first rule. But if we apply the first rule to $f(f(c))$ we get $f(f(d))$, which cannot be further rewritten because sort information has been lost!

Checking Sort-Decreasingness

Sort decreasingness can be easily checked, since we do not need to check it on the (infinite) set of all substitutions θ. If $\{x_1 : s_1, \ldots, x_n : s_n\} = vars(t \to t')$, we only need to check it on the finite set of substitutions of the form $\{x_1 : s_1 \mapsto x'_1 : s'_1, \ldots, x_n : s_n \mapsto x'_n : s'_n\}$, $s'_i \leq s_i$, $1 \leq i \leq n$, called the sort specializations of the variables $\{x_1 : s_1, \ldots, x_n : s_n\}$.

For example, for sorts $Nat < Set$, with $\_ \cup \_$ set union, the rule $x \to x \cup x$, with $x : Set$, is not sort-decreasing, since for the sort specialization $\{x : Set \mapsto x' : Nat\}$ we have $ls(x') = Nat < Set = ls(x' \cup x')$.

Exercise. For Σ preregular, prove that the rules $\vec{E}$ are sort decreasing iff for each $t \to t'$ in $\vec{E}$ and for each sort specialization ρ (a fine number of ρ's if $(S, <)$ is finite) we have: $ls(t\rho) \geq ls(t'\rho)$.

B-Preregular Signatures

Recall that if Σ is preregular each term t has a least sort $ls(t)$. For axioms B we want the strongest property: that Σ is B-preregular, i.e., (1) Σ is preregular, and (2) $t =_B t'$ implies $ls(t) = ls(t')$.

How can we check that Σ is B-preregular? Very easily. Assume all axioms B are such that for each $(u = v) \in B$ we have $vars(u) = vars(v)$. Now consider the following:

Theorem

For Σ preregular and Σ-axioms B as above, Σ is B-preregular iff the rewrite theory $(\Sigma, \overrightarrow{B} \cup \overleftarrow{B})$ is sort decreasing.

The theorem follows easily from the properties of sort-decreasing rules stated in previous slides. Furthermore, it can be effectively checked for each $(u = v) \in B$ using its sort specializations. Maude automatically checks that Σ is B-preregular and gives a warning if the property fails.

The Algebra $\mathbb{T}_{\Sigma/B}$

For Σ B-preregular we can easily define the algebra $\mathbb{T}_{\Sigma/B}$, whose elements are B-equivalence classes $[t]_B$ of terms modulo $=_B$, i.e., $t' \in [t]_B \;\Leftrightarrow\; t =_B t'$. Specifically, $\mathbb{T}_{\Sigma/B} = (T_{\Sigma/B}, \_{\mathbb{T}_{\Sigma/B}})$, where, abbreviating $[t]_B$ to just $[t]$, we define:

- $T_{\Sigma/B} = \{T_{\Sigma,s}/{=_B}\}_{s \in S}$, with $T_{\Sigma,s}/{=_B}$ written $T_{\Sigma/B,s}$.
- For $a :\to s$ in Σ, $a_{\mathbb{T}_{\Sigma/B}} = [a] \in T_{\Sigma/B,s}$.
- For $f : s_1 \ldots s_n \to s$ in Σ, $f_{\mathbb{T}_{\Sigma/B}} : T_{\Sigma/B,s_1} \times \ldots \times T_{\Sigma/B,s_n} \ni ([t_1], \ldots, [t_n]) \mapsto [f(t_1, \ldots, t_n)] \in T_{\Sigma/B,s}$.

Note that the definition of $f_{\mathbb{T}_{\Sigma/B}}$ does not depend on the choice of the $t_1, \ldots, t_n$, since if $t_i =_B t_i'$, $1 \leq i \leq n$, then we have: $f(t_1, \ldots, t_n) =_B f(t_1', \ldots, t_n')$, since we can build a proof from the proofs of $t_i =_B t_i'$, $1 \leq i \leq n$.

Determinism

Another requirement on $(\Sigma, B, \vec{E})$ is determinism: if a term t is simplified by $\vec{E}$ modulo B to two different terms u and v, and $u \neq_B v$, then u and v can always be further simplified by $\vec{E}$ modulo B to a common term w.

This implies (Exercise!) that if $t \rightarrow^{\star}_{\vec{E}/B} u$ and $t \rightarrow^{\star}_{\vec{E}/B} v$, and u and v cannot be further simplified by $\vec{E}$ modulo B, then we must have $u =_B v$. This is the idea of determinism: if rewriting with $\vec{E}$ modulo B yields a fully simplified answer, then that answer must be unique modulo B.

That is, modulo the axioms B, the final result of rewriting a term t with the rules $\vec{E}$ modulo B should not depend on the particular order in which the rewrites have been performed.

Determinism = Confluence

Determinism is captured by confluence. The rules $\vec{E}$ of $(\Sigma, B, \vec{E})$ are confluent modulo B iff for each $t \in T^{\circ}_{\Sigma(Y)}$, whenever $t \rightarrow^{\star}_{\vec{E}/B} u$, $t \rightarrow^{\star}_{\vec{E}/B} v$, there is a $w \in T^{\circ}_{\Sigma(Y)}$ such that $u \rightarrow^{\star}_{\vec{E}/B} w$ and $v \rightarrow^{\star}_{\vec{E}/B} w$. This can be described diagrammatically (dashed arrows denote existential quantification):

$$
\begin{tikzcd}
 & t \arrow[dl, "\star"', "\vec{E}/B"] \arrow[dr, "\star", "\vec{E}/B"'] & \\
u \arrow[dr, dashed, "\star"', "\vec{E}/B"] & & v \arrow[dl, dashed, "\star", "\vec{E}/B"'] \\
 & w &
\end{tikzcd}
$$

$\vec{E}$ is ground confluent modulo B if this holds for all $t \in T^{\circ}_{\Sigma}$.

Termination

Definition

For the rewrite theory $(\Sigma, B, \vec{E})$, rules $\vec{E}$ are called terminating modulo B iff $\rightarrow_{\vec{E}/B}$ is well-founded. $\vec{E}$ is called weakly terminating modulo B iff any $t \in T^{\circ}_{\Sigma(Y)}$ has a $\vec{E}/B$-normal form, i.e., $\exists v \in T^{\circ}_{\Sigma(Y)}$ s.t. $t \rightarrow^{\star}_{\vec{E}/B} v \wedge \nexists w \in T^{\circ}_{\Sigma(Y)}$ s.t. $v \rightarrow_{\vec{E}/B} w$. (**Notation:** $t \rightarrow^{!}_{\vec{E}/B} v$).

If $(\Sigma, B, \vec{E})$ is ground confluent and terminating modulo B, each $t \in T_{\Sigma}$ reduces to an $\vec{E}/B$-normal form denoted $t!_{\vec{E}/B}$, and $t!_{\vec{E}/B}$ is unique modulo B, i.e., $[t!_{\vec{E}/B}]$ is unique. Furthermore, if Σ is B-preregular, and $\vec{E}$ is sort-decreasing, both **Unique Termination** and **Sort Preservation** hold, and we have an S-sorted function:

$$\_!_{\vec{E}/B} : T_{\Sigma} \ni t \mapsto [t!_{\vec{E}/B}] \in T_{\Sigma/B}$$

Joinability and the Church-Rosser Property

Call two terms $t, t' \in T^{\circ}_{\Sigma(Y)}$ joinable with $\vec{E}$ modulo B, denoted $t \downarrow_{\vec{E}/B} t'$, iff $(\exists w \in T^{\circ}_{\Sigma(Y)})\ t \rightarrow^{\star}_{\vec{E}/B} w \wedge t' \rightarrow^{\star}_{\vec{E}/B} w$.

Execise. Prove that if $(\Sigma, E \cup B)$ is an order-sorted equational theory whose rules $\vec{E}$ are confluent modulo B, then the following equivalence, called the Church-Rosser property, holds for any two terms $t, t' \in T^{\circ}_{\Sigma(Y)}$:

$$(\dagger) \quad t =_{E \cup B} t' \ \Leftrightarrow\ t \downarrow_{\vec{E}/B} t'.$$

Prove that if $\vec{E}$ is also terminating modulo B we also have:

$$(\ddagger) \quad t =_{E \cup B} t' \ \Leftrightarrow\ t!_{\vec{E}/B} =_B t'!_{\vec{E}/B}$$

Since $=_B$ (with B A, C, U axioms) is decidable, we can decide $t =_{E \cup B} t'$ by deciding $t!_{\vec{E}/B} =_B t'!_{\vec{E}/B}$, which we can do in Maude by typing: `red t == t' .` ($\dagger$) reduces equational deduction to rewriting, and ($\ddagger$) makes it decidable.

Subsignatures and Constructor Subsignatures

Before defining sufficient completeness we make more precise the notions of subsignature and constructor subsignature.

Definition

An order-sorted signature $\Sigma' = ((S', <'), F', G'), \Sigma'$ is called a subsignature of an order-sorted signature $\Sigma = ((S, <), F, G)$, denoted $\Sigma' \subseteq \Sigma$, iff:

1. $S' \subseteq S$, $<' \subseteq <$, and $F' \subseteq F$.
2. $G' \subseteq G$, i.e., for each $(f' : w' \to s') \in G'$ we have $(f' : w' \to s') \in G$.

If $S' = S$ and $<' = <$ we say that $\Sigma' \subseteq \Sigma$ on the same sort poset.

In a functional module `fmod` $(\Sigma, E \cup B)$ `endfm`, the `ctor` declaration defines a subsignature $\Omega \subseteq \Sigma$ on the same sort poset $(S, <)$, called the constructor subsignature.

Sufficient Completeness Defined

Definition

Let the rewrite theory $(\Sigma, B, \vec{E})$ be terminating, and $\Omega \subseteq \Sigma$ a subsignature inclusion, where Ω has the same poset of sorts as Σ. We call the rules $\vec{E}$ sufficiently complete modulo B with respect to the constructor subsignature Ω iff for each $t \in T_\Sigma$ and each $\vec{E}/B$-normal form of t, i.e., each $u \in T_\Sigma$ s.t. $t \to^{!}_{\vec{E}/B} u$, we have $u \in T_\Omega$.

More on Sufficient Completeness

If $\Sigma^{\Box}$ is kind-complete, then the above requirement that for each $t \in T_{\Sigma}$, if $t \to!_{\vec{E}/B} u$ then $u \in T_{\Omega}$ should apply only to $t \in T_{\Sigma,s}$ with $s \in S$ in the original set of sorts, before adding the "kind" $[s]$ on top of each connected component $[s]$ and lifting operators to kinds. I.e., the sufficient completeness for $\vec{E}$ modulo B should be required only for terms in the original signature Σ before kind-completing it to $\Sigma^{\Box}$.

Example. For sorts Nat and $NzNat$ with $Nat < NzNat$, and constructors $0 :\to Nat$ and $s : Nat \to NzNat$, the predecessor function $p : NzNat \to Nat$ defined by the equation $p(s(x)) = x$ is sufficiently complete. But the term $p(0)$ of kind $[Nat]$ is in normal form, yet is not a constructor term.

More on Sufficient Completeness (II)

If $(\Sigma, B, \vec{E})$ has $\Omega \subseteq \Sigma$ as a constructor subsignature with $\vec{E}$ terminating modulo B, we say that the constructors Ω are free modulo B in $(\Sigma, B, \vec{E})$ iff for each sort s which is not a kind and each $u \in T_{\Omega,s}$ we have $u = u!_{\vec{E}/B}$. That is, each $u \in T_{\Omega,s}$ is in $\vec{E}, B$-normal form.

Example. Multisets of natural numbers, with $Nat < MSet$, and constructors $\emptyset : \to MSet$ and $\_,\_ : MSet\ MSet \to MSet$ and axioms ACU for $\_,\_$ are free modulo ACU. But Sets of natural numbers, obtained by adding the equation $n, n = n$, where n has sort Nat are not free modulo ACU. For example, the set $0, 0, s(0)$ is not in $\vec{E}, B$-normal form, since $(0, 0, s(0))!_{\vec{E}/ACU} = 0, s(0)$.

The Canonical Term Algebra

Let fmod $(\Sigma, E \cup B)$ enfm have Σ B-preregular and constructor subsignature $\Omega \subseteq \Sigma$; and let $(\Sigma, B, \vec{E})$ be sort-decreasing, confluent, terminating and sufficiently complete modulo B (w.r.t. Ω). Then, the semantics of fmod $(\Sigma, E \cup B)$ enfm is defined by its canonical term algebra $\mathbb{C}_{\Sigma/E,B} = (C_{\Sigma/E,B}, \_{\mathbb{C}_{\Sigma/E,B}})$, where:

- for each $s \in S$, $C_{\Sigma/E,B,s} = \{[u] \in T_{\Omega/B,s} \mid u = u!_{\vec{E}/B}\}$
- For $a : \to s$ in Σ, $a_{\mathbb{C}_{\Sigma/E,B}} = [a!_{\vec{E}/B}] \in C_{\Sigma/E,B,s}$.
- For $f : s_1 \ldots s_n \to s$ in Σ,
  $f_{\mathbb{C}_{\Sigma/E,B}} : C_{\Sigma/E,B,s_1} \times \ldots \times C_{\Sigma/E,B,s_n} \ni ([t_1], \ldots, [t_n]) \mapsto [f(t_1, \ldots, t_n)!_{\vec{E}/B}] \in C_{\Sigma/E,B,s}$.

Ground confluence and termination imply **Unique Termination**. **Sufficient Completeness** is guaranteed. Sort-decreasingness and B-preregularity imply **Sort Preservation**. $\mathbb{C}_{\Sigma/E,B}$ now allows: (i) axioms B, and (ii) constructors that need not be free modulo B.

Example of Canonical Term Algebra

Consider the following:

- A signature Ω of constructors with sorts Nat and Set, subsort $Nat < Set$, and constructors $0 :\rightarrow Nat$, $s : Nat \rightarrow Nat$, $\emptyset :\rightarrow Set$ and $\_,\_ : Set\ Set \rightarrow Set$ and axioms $B = ACU$ for $\_,\_$.
- Σ adds to Ω the function symbol $+1 : Set \rightarrow Set$.
- $E = \{(n, n) = n,\ +1(\emptyset) = \emptyset,\ +1(n, S) = s(n), +1(S)\}$, where n has sort Nat and S has sort Set.

Then, up to the slight change of representation $n_1, \ldots, n_k$ versus $\{n_1, \ldots, n_k\}$, $\mathbb{C}_{\Sigma/E,B}$ is the algebra with sorts Nat, resp. Set, interpreted as $\mathbb{N}$, resp. $\mathcal{P}_{fin}(\mathbb{N})$, set union function, denoted $\_,\_{}_{\mathbb{C}_{\Sigma/E,B}}$, and the function $+1_{\mathbb{C}_{\Sigma/E,B}}$ increases by 1 each set element.

Examples of Sufficient Completeness Modulo B

For example, consider the reverse function in the list module

```
fmod MY-LIST is protecting NAT .
  sorts NeList List .
  subsorts Nat < NeList < List .
  op _;_ : List List -> List [assoc] .
  op _;_ : NeList NeList -> NeList [assoc ctor] .
  op nil : -> List [ctor] .
  op rev : List -> List .
  eq rev(nil) = nil .
  eq rev(N:Nat ; L:List) = rev(L:List) ; N:Nat .
endfm
```

Are `nil` and $\_;\_$ (plus `0` and `s`) really the constructors of this module as claimed?

Examples of Sufficient Completeness Modulo B (II)

The answer is that they are not, as witnessed by:

```
Maude> red rev(7) .
reduce in MY-LIST : rev(7) .
rewrites: 0 in 0ms cpu (0ms real) (~ rewrites/second)
result List: rev(7)
```

The problem is that the above two equations would have been sufficient if we had also declared the `id: nil` attribute for `_;_` but do not fully define `rev` if only the `assoc` attribute is used.

In future lectures we shall see how sufficient completness can be automatically checked under reasonable assumptions.

Examples of Sufficient Completeness Modulo B (III)

So, suppose we add an extra equation for `rev`

```
fmod MY-LIST is protecting NAT .
  sorts NeList List .
  subsorts Nat < NeList < List .
  op _;_ : List List -> List [assoc] .
  op _;_ : NeList NeList -> NeList [assoc ctor] .
  op nil : -> List [ctor] .
  op rev : List -> List .
  eq rev(nil) = nil .
  eq rev(N:Nat) = N:Nat .
  eq rev(N:Nat ; L:List) = rev(L:List) ; N:Nat .
endfm
```

Is now this module sufficiently complete?

Examples of Sufficient Completeness Modulo B (IV)

Indeed we now have

```
Maude> red rev(7) .
reduce in MY-LIS
```

But it is still not sufficiently complete, since

```
Maude> red nil ; 7 .
reduce in MY-LIST : nil ; 7 .
result List: nil ; 7
```

is not a constructor term, since _;_ is a constructor on `NeList` but a defined function on `List`.

Examples of Sufficient Completeness Modulo B (V)

The really sufficiently complete specification, making the constructors free modulo assoc, is

```
fmod MY-LIST is protecting NAT .   sorts NeList List .
  subsorts Nat < NeList < List .
  op _;_ : List List -> List [assoc] .
  op _;_ : NeList NeList -> NeList [assoc ctor] .
  op nil : -> List [ctor] .
  op rev : List -> List .
  eq rev(nil) = nil .
  eq rev(N:Nat) = N:Nat .
  eq rev(N:Nat ; L:List) = rev(L:List) ; N:Nat .
  eq nil ; L:List = L:List .
  eq L:List ; nil = L:List .
endfm

Maude> red nil ; 7 .
reduce in MY-LIST : nil ; 7 .
result NzNat: 7
```

Examples of Sufficient Completeness Modulo B (VI)

Sets of natural numbers do not have free constructors. The following is another example of an executable equational theory whose constructors are not free.

```
fmod  NAT/3 is
  sorts Nat .
  op 0 : -> Nat [ctor] .
  op s : Nat -> Nat [ctor] .
  op _+_ : Nat Nat -> Nat .
  vars N M : Nat .
  eq N + 0 = N .
  eq N + s(M) = s(N + M) .
  eq s(s(s(0))) = 0 .
endfm
```