Program Verification: Lecture 24

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Constructor Pattern Predicates

Recall from Lecture 18 that for an executable rewrite theory $\mathcal{R} = (\Sigma, E \cup B, R)$ with constructor subsignature Ω and state sort St, an expressive set Π of state predicate names to specify modal properties of $\mathbb{C}_{\mathcal{R}}$ is the set of constrained constructor patterns $u|\varphi$,

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DNF Theorem. Any $p \in PCPattF$ has a disjunctive normal form, $dnf(p)$, which is either $\bot$ or has the form $u_1 \vee \ldots \vee u_n$, with $u_i \in T_\Omega(X)_{St}$, $1 \leq i \leq n$, $n \geq 1$, and is such that $[\![p]\!] = [\![dnf(p)]\!]$.

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NCNF Theorem. Any $n \in NCPattF$ has a negative conjunctive normal form, $ncnf(n)$, with $ncnf(n)$ either $\top$ or of the form $\neg u_1 \wedge \ldots \wedge \neg u_n$, $u_i \in T_{\Omega}(X)_{St}$, $1 \leq i \leq n$, $n \geq 1$, and s.t. $[\![n]\!] = [\![ncnf(n)]\!]$.

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By definition, given positive pattern formulas $u_1 \vee \ldots \vee u_n$ and $v_1 \vee \ldots \vee v_m$, $n, m \geq 1$, (i) $\bot \sqsubseteq_B u_1 \vee \ldots \vee u_n$, and (ii) $u_1 \vee \ldots \vee u_n$ is B-subsumed by $v_1 \vee \ldots \vee v_m$, denoted $u_1 \vee \ldots \vee u_n \sqsubseteq_B v_1 \vee \ldots \vee v_m$, iff $\forall i$, $1 \leq i \leq n$, $\exists j$, $1 \leq j \leq n$, s.t., $u_i \sqsubseteq_B v_j$. Obviously, for B any combination of A and/or C and/or U axioms, the relation $u_1 \vee \ldots \vee u_n \sqsubseteq_B v_1 \vee \ldots \vee v_m$ is decidable.

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Reachable, Transition-Closed, and Inductive Invariants

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Ex.24.1: Prove that: (1) Q is $\mathcal{R}$-transition-closed iff $\mathcal{R}^*[Q] = Q$, and (2) the smallest invariant from a set of initial states $I \subseteq T_{\Omega/B,St}$, namely, $\mathcal{R}^*[I]$, is inductive.

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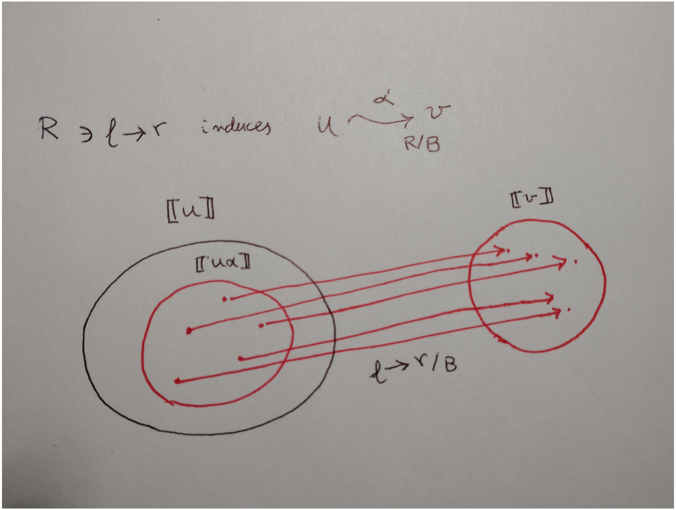
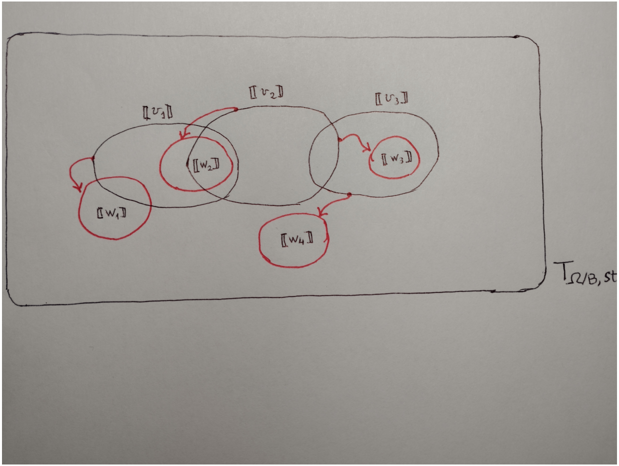
The Set-Theoretic Meaning of Narrowing



The Set-Theoretic Meaning of Folding Narrowing

For $F_d = v_1 \vee v_2 \vee v_3$, then $F_{d+1} = w_1 \vee w_2 \vee w_4$. w_3 folded into v_3.



Completeness of Folding Narrowing

Completeness Theorem of Folding Narrowing. Let (Ω, B, R) be a topmost rewrite theory with state sort St, and $u_1 \vee \ldots \vee u_n$ an initital state. For each depth $d \in \mathbb{N}$, $[\![P_d]\!] = \mathcal{R}^{\leq d}[\![u_1 \vee \ldots \vee u_n]\!]$.

If it exists, let d be the smallest depth such that $F_{d+1} = \bot$. Then, $P_{d+1} = P_d \vee F_{d+1} = P_d \vee \bot$, which implies $[\![P_d]\!] = [\![P_{d+1}]\!]$. I.e., $\mathcal{R}^{\leq d}[\![u_1 \vee \ldots \vee u_n]\!] = [\![P_d]\!] = [\![P_{d+1}]\!] = \mathcal{R}^{\leq d+1}[\![u_1 \vee \ldots \vee u_n]\!] = \mathcal{R}[\mathcal{R}^{\leq d}[\![u_1 \vee \ldots \vee u_n]\!]] \cup \mathcal{R}^{\leq d}[\![u_1 \vee \ldots \vee u_n]\!]$, so that $[\![P_d]\!]$ is transition-closed. Therefore, by **Ex.24.1** we have $[\![P_d]\!] = \mathcal{R}^*[\![P_d]\!]$. But then $[\![P_d]\!] = \mathcal{R}^*[\![u_1 \vee \ldots \vee u_n]\!]$ follows from the inclusions:

$$\mathcal{R}^*[\![u_1 \vee \ldots \vee u_n]\!] \subseteq \mathcal{R}^*[\![P_d]\!] = [\![P_d]\!] \subseteq \mathcal{R}^*[\![u_1 \vee \ldots \vee u_n]\!].$$

That is, we get a finite, symbolic descrition of all reachable states $\mathcal{R}^*[\![u_1 \vee \ldots \vee u_n]\!]$ as the pattern disjunction P_d.

Four Methods to Symbolically Verify Invariants

For $\mathcal{R} = (\Omega, B, R)$ a topmost rewrite theory with state sort St, $u_1 \vee \ldots \vee u_n$ an inititial state, and $Q \subseteq T_{\Omega/B,St}$, the following four methods can verify (†) $\mathbb{C}_{\mathcal{R}}, [\![u_1 \vee \ldots \vee u_n]\!] \models_{S4} \Box Q$.

A. If Q is specifiable as $Q = [\![n]\!]$ for n a negative pattern formula different from $\top$ (if $n = \top$, (†) holds trivially). W.L.O.G. we may assume $n = ncnf(n) = \neg v_1 \wedge \ldots \wedge \neg v_m$.

Method 1. (†) holds if $\mathbb{C}_{\mathcal{R}}, [\![u_1 \vee \ldots \vee u_n]\!] \not\models_{S4} \Diamond[\![v_1 \vee \ldots \vee v_m]\!]$. A sufficient condition to automatically verify (†) is that the m commands `{fold} vu-narrow` $u_1 \vee \ldots \vee u_n$ `=>*` v_j, $1 \leq j \leq m$ return: `No solution`.

If this succeeds, Maude can retun the positive pattern disjunction P_d such that $[\![P_d]\!] = \mathcal{R}^*[\![u_1 \vee \ldots \vee u_n]\!]$, which enables **Method 2**.

Four Methods to Symbolically Verify Invariants (II)

Method 2. If we have found $P_d = w_1 \vee \ldots \vee w_k$ s.t. $[\![P_d]\!] = \mathcal{R}^*[\![u_1 \vee \ldots \vee u_n]\!]$, then (†) holds for any Q of the form, $Q = [\![\neg v_1 \wedge \ldots \wedge \neg v_m]\!]$ iff $\forall 1 \leq i \leq k, \forall 1 \leq j \leq m, w_i \wedge v_j = \bot$, i.e., (see Appendix 1), iff $Unif_B(w_i = v_j) = \emptyset$ for all i, j (we assume $vars(w_i) = vars(v_j)$). Note that no search is needed!

B. If Q is specifiable as $Q = [\![p]\!]$ for p a positive pattern formula different from $\bot$ (if $p = \bot$, (†) cannot hold). W.L.O.G. we may assume $p = dnf(p) = v_1 \vee \ldots \vee v_m$.

Method 3. If we have found $P_d = w_1 \vee \ldots \vee w_k$ s.t. $[\![P_d]\!] = \mathcal{R}^*[\![u_1 \vee \ldots \vee u_n]\!]$, then (†) holds for any Q of the form, $Q = [\![v_1 \vee \ldots \vee v_m]\!]$ iff $w_1 \vee \ldots \vee w_k \subseteq_B v_1 \vee \ldots \vee v_m$. A decidable sufficient condition is $w_1 \vee \ldots \vee w_k \sqsubseteq_B v_1 \vee \ldots \vee v_m$.

Four Methods to Symbolically Verify Invariants (III)

Method 4. (†) holds for $Q = [\![v_1 \vee \ldots \vee v_m]\!]$ if: (1) Q is transitition-closed; this holds iff a @fold vu-narrow $v_1 \vee \ldots \vee v_m$ =>1 $ command, where $ is a fresh (and therefore unreachable) constant added to $\mathcal{R}$, generates an $F_1(v_1 \vee \ldots \vee v_m)$ s.t. either $F_1(v_1 \vee \ldots \vee v_m) = \bot$, or $F_1(v_1 \vee \ldots \vee v_m) \subset_B v_1 \vee \ldots \vee v_m$. (2) $u_1 \vee \ldots \vee u_n \subset_B v_1 \vee \ldots \vee v_m$. A decidable sufficient condition is $u_1 \vee \ldots \vee u_n \sqsubset_B v_1 \vee \ldots \vee v_m$.