Appendix 1 to Lecture 24

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Before proving the DNF and NCNF theorems for positve (res. negative) pattern formulas, note that the sets $T_{\Omega}(X)_{St}$, *PCPattF*, and *NCPattF* are subsets of the set $T_{\Sigma_{BL}}(T_{\Omega}(X)_{St})$ of Boolean expressions with constants $T_{\Omega}(X)_{St}$ for the unsorted signature $\Sigma_{BL} = \{\vee, \wedge, \neg, \top, \bot\}$. That is, they are subsets of the underlying set of the *free* Σ_{BL}-algebra $\mathbb{T}_{\Sigma_{BL}}(T_{\Omega}(X)_{St})$ when we regard the set of constants $T_{\Omega}(X)_{St}$ as the set of "variables" of such a free algebra. Note also that the powerset $\mathcal{P}(T_{\Omega/B,St})$ is a Σ_{BL}-algebra with $\vee$ interpreted as $\cup$, $\wedge$ interpreted as $\cap$, $\neg$ interpreted as $\lambda U \in \mathcal{P}(T_{\Omega/B,St}).\ T_{\Omega/B,St} \backslash U$, $\top$ interpreted $T_{\Omega/B,St}$, and $\bot$ interpreted as $\emptyset$. Furthemore, $\mathcal{P}(T_{\Omega/B,St})$ is a *Boolean algebra*, i.e., $\mathcal{P}(T_{\Omega/B,St}) \models E_{BL}$, where E_{BL} are the equations axiomatizing Boolean algebras.

Therefore, by the Freeness Theorem, the assignment map $[\![\_]\!] : T_{\Omega}(X)_{St} \rightarrow \mathcal{P}(T_{\Omega/B,St})$ defining the semantics of constructor patterns extends to a unique Σ_{BL}-homomorphism, also denoted $[\![\_]\!]$, of the form:

$$[\![\_]\!] : \mathbb{T}_{\Sigma_{BL}}(T_{\Omega}(X)_{St}) \rightarrow \mathcal{P}(T_{\Omega/B,St})$$

It then follows immediately by the Completeness Theorem for equational Logic and $\mathcal{P}(T_{\Omega/B,St})$ being a Boolean algebra, that for any two Boolean expressions $q, q' \in T_{\Sigma_{BL}}(T_{\Omega}(X)_{St})$ we have the implication:

$$E_{BL} \vdash q = q' \quad \Rightarrow \quad [\![q]\!] = [\![q']\!].$$

Therefore, for any $p, p' \in$ *PCPattF* and $n, n' \in$ *NCPattF* we have:

$$E_{BL} \vdash p = p' \quad \Rightarrow \quad [\![p]\!] = [\![p']\!] \qquad \textit{and} \qquad E_{BL} \vdash n = n' \quad \Rightarrow \quad [\![n]\!] = [\![n']\!].$$

In particular, using the distributivity equation $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ in E_{BL} as a terminating rewrite rule modulo AC, any positive formula p is E_{BL}-equal to a formula p in *disjunctive normal form*, i.e., p' is either $\bot$, or u, or has the form

$$(\dagger) \quad p' = \bigvee_{i \in I} \bigwedge_{j \in J_i} u_j$$

i.e., it is a disjunction of conjunctions of patterns.

The next observation is that for each positive pattern formula $p_1 = \bigwedge_{j \in J} u_j$ there is another positive pattern fomula p'_1 with $[\![p_1]\!] = [\![p'_1]\!]$, and with p'_1 of the form $\bigvee_{k \in K} v_j$, where, by convention, when $K = \emptyset$, then $p'_1 = \bot$, and if K is a singleton set, then p'_1 is a single pattern. Why is this so?

First of all note also that the semantic function $[\![\_]\!] : T_{\Omega}(X)_{St} \rightarrow \mathcal{P}(T_{\Omega/B,St})$ enjoys the property that for any sort-preserving and bijective substitution $\alpha : X \rightarrow X$ (called a *variable renaming*) we trivially have the identity $[\![u]\!] = [\![u\alpha]\!]$. This means that for any conjunction $v_1 \wedge \ldots v_n$ W.L.O.G. we may assume that $vars(v_i) \cap vars(v_j) = \emptyset$, $1 \leqslant i < j \leqslant n$, since otherwise we can always rename the variables of each v_i by a variable renaming α_i so that $vars(v_i\alpha_i) \cap vars(v_j\alpha_j) = \emptyset$ holds. Second, assuming the above variable disjointness, we have:

$$(\ddagger) \quad [\![v_1 \wedge \ldots \wedge v_n]\!] \;=\; \bigcup_{\alpha \in Unif_B(v_1=v_2 \wedge \ldots \wedge v_1=v_n)} [\![v_1\alpha]\!] \;=\; [\![\bigvee_{\alpha \in Unif_B(v_1=v_2 \wedge \ldots \wedge v_1=v_n)} v_1\alpha]\!]$$

That is, as claimed above, any conjunction of patterns is semantically equivalent to a disjunction of patterns. The above identities in fact hold because, by definition of $[\![\_]\!]$, $[\![v_1 \wedge \ldots \wedge v_n]\!] = [\![v_1]\!] \cap \ldots \cap [\![v_n]\!]$. But $[w] \in [\![v_1]\!] \cap \ldots \cap [\![v_n]\!]$ iff there are ground substitutions $\rho_1, \ldots \rho_n$ such that: (i) $[w] = [v_1\rho_1]$, and (ii) $v_1\rho_1 =_B v_2\rho_2 \wedge \ldots v_1\rho_1 =_B v_n\rho_n$, that is, iff $\rho_1 \uplus \ldots \uplus \rho_n$ is a B-unifier of the system of equations $v_1 = v_2 \wedge \ldots v_1 = v_n$. But this is the case iff there is a unifier $\alpha \in Unif_B(v_1 = v_2 \wedge \ldots \wedge v_1 = v_n)$ and a ground unifier τ such that $\rho_1 \uplus \ldots \uplus \rho_n =_B \alpha\tau$. Of course, if $Unif_B(v_1 = v_2 \wedge \ldots \wedge v_1 = v_n) = \emptyset$, then $v_1 \wedge \ldots \wedge v_n$ is semantically equivalent to $\bot$. Therefore, the identities ($\ddagger$) hold as claimed. Then it immediately follows from ($\dagger$) and ($\ddagger$) that, as claimed, any possitive pattern formula is semantically equivalent to either $\bot$ or to a pattern disjunction. That is, we then obtain as a trivial corollary the DNF Theorem,

DNF Theorem. Any $p \in PCPattF$ has a *disjunctive normal form*, $dnf(p)$, which is either $\bot$ or has the form $u_1 \vee \ldots \vee u_n$, with $u_i \in T_\Omega(X)_{St}$, $1 \leqslant i \leqslant n$, $n \geqslant 1$, and is such that $[\![p]\!] = [\![dnf(p)]\!]$. □

We are now ready to prove the CNF Theorem,

NCNF Theorem. Any $n \in NCPattF$ has a *negative conjunctive normal form*, $ncnf(n)$, which is either $\top$ or has the form $\neg u_1 \wedge \ldots \wedge \neg u_n$, with $u_i \in T_\Omega(X)_{St}$, $1 \leqslant i \leqslant n$, $n \geqslant 1$, and is s.t. $[\![n]\!] = [\![ncnf(n)]\!]$. Note that $[\![\neg u_1 \wedge \ldots \wedge \neg u_n]\!] = T_{\Omega/B,St} \backslash [\![u_1 \vee \ldots \vee u_n]\!]$.

Proof: E_{BL} includes the De Morgan Laws $\neg x \wedge \neg y = \neg(x \vee y)$ and $\neg x \vee \neg y = \neg(x \wedge y)$. Using them as left-to-right rewrite rules modulo AC, we can put any negative pattern formula n into a semantically equivalent form $\neg(p)$, where p is a positive pattern formula. Furthermore, by the DNF Theorem we can put p itself in DNF form, i.e., $\neg(p)$ is semantically equivalent to a Boolean formula of either the form $\neg(\bot)$ or the form $\neg(u_1 \vee \ldots \vee u_n)$. But then, $\neg(\bot) = \top$ is in E_{BL}, and using $\neg x \vee \neg y = \neg(x \wedge y)$ as a right-to-left rewrite rule modulo AC, $\neg(u_1 \vee \ldots \vee u_n)$ is semantically equivalent to the negative pattern formula $\neg u_1 \wedge \ldots \wedge \neg u_n$, $n \geqslant 1$, as desired. □

A last, useful propoperty about the semantic function $u \mapsto [\![u]\!]$, is that, it follows easily from the definition of $[\![u]\!]$ that $[\![u\alpha]\!] \subseteq [\![u]\!]$ for any substitution α.