Program Verification: Lecture 23

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A3: By unifying $x + y$ with the lefthand sides $n + 0$ and $n + s(m)$ equations [1], [2]. This gives unifiers $\theta_1 = \{n \mapsto x, y \mapsto 0\}$, which evaluates to y with rule [1], and $\theta_2 = \{n \mapsto x, y \mapsto s(y'), m \mapsto y'\}$, which evaluates to $s(x + y')$ with rule [2].

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Note that if $t \in T_\Sigma$, $t \to_R t[r\theta]_p$ iff $t \overset{\theta}{\rightsquigarrow}_R t[r]_p\theta$. I.e., narrowing and rewriting coincide for ground terms.

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As for rewriting, we have the reflexive-transitive closure $t \overset{\theta}{\leadsto}{}^{*}_{R} v$, where for 0 steps we get $\theta = id$ and $v = t$, and for $n + 1$ steps we get a sequence:

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with $v = t_{n+1}$ and θ the composed substitution $\theta = \theta_1 \ldots \theta_{n+1}$. To avoid variable capture, we always assume that rules in R are variable renamed so that they do not share any variables with any of the terms t_i; and that for each unifier θ_i, $1 \leq i \leq n + 1$, the variables in $rng(\theta_i) = \{y \in X \mid \exists x \in dom(\theta_i)\ s.t.\ y \in vars(\theta_i(x)\}$, are fresh (i.e., never used before).

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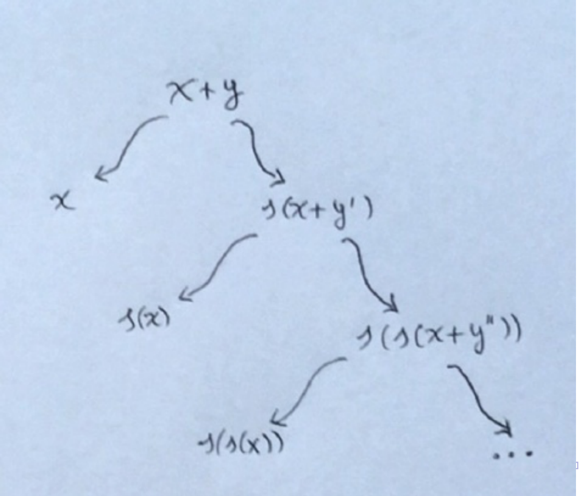

The Lifting Lemma

Symbolic computation by narrowing covers all rewriting computations as instances as shown below (proof in Appendix):

Theorem (Lifting Lemma). Let (Σ, R) be a term rewriting system, $t \in T_{\Sigma}(X)$, and θ an R-irreducible substitution (i.e., if $x \in dom(\theta)$, then $\theta(x)$ cannot be rewritten with R).

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Since each narrowing step in the Lifting Lemma preserves the invariant that the substitution θ for t, resp. γ for v, is R-irreducible, the Lifting Lemma extends in a straightforward manner to R-rewriting sequences of the form $t\theta \to_R^* w$ with θ an R-irreducible, which are indeed all covered as instances by narrowing sequences $t \overset{\theta_1}{\leadsto}_R t_1 \dots t_n \overset{\theta_{n+1}}{\leadsto}_R t_{n+1}$, with $w = t_{n+1}\delta$.

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A small technicality is that we should narrow t not just with R, but with all its B-extensions, which for R/B-rewriting is done automatically by Maude (see §4.8 in "All About Maude").

Topmost Rewrite Theories

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Many rewrite theories can be easily transformed into semantically equivalent topmost ones. For example, if $\mathcal{R}$ specifies a concurrent object system, we can just add a new sort *State* and a constructor $\{\_\} : Configuration \to State$ and convert, for example, a rule $credit(O, M)\ \langle O : Accnt \mid bal : N \rangle \to \langle O : Accnt \mid bal : N + M \rangle$ into the semantically equivalent rule:

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The proof is a simple application of the Lifting Lemma and is left as an exercise.

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The only requirement is that the negation of the invariant (i.e., its complement) can be expressed as a term u with variables, or, more generally, as a finite set $\{u_1, \ldots, u_m\}$ of terms with variables.

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Folding can make the search graph finite, allowing termination of narrowing seach and making verification of an invariant decidable.

Narrowing with Folding May Terminate

Just as for the `search` command, a narrowing search may not terminate. However, Maude supports a `{fold} vu-narrow` narrowing search command option that tries to fold the usually infinite narrowing search tree into a hopefully finite narrowing search graph, by not exploring tree nodes that are substitution instances modulo B of more general, previously explored nodes.

Definition. Term u, reached by narrowing at depth d, is folded into another term v reached by narrowing at depth $\leq d$ (i.e., u is no further explored) if there is a substitution α s.t. $u =_B v\alpha$.

Folding can make the search graph finite, allowing termination of narrowing seach and making verification of an invariant decidable.

Lets us see an example. Consider the following Maude specification of Lamport's bakery protocol:

Lamport's Bakery Protocol

```
mod BAKERY is sorts Nat LNat Nat? State WProcs Procs .
  subsorts Nat LNat < Nat? .  subsort WProcs < Procs .
  op 0 : -> Nat .                    op s : Nat -> Nat .
  op [_] : Nat -> LNat .                     *** number-locking operator
  op < wait,_> : Nat -> WProcs .    op < crit,_> : Nat -> Procs .
  op mt : -> WProcs .                 *** empty multiset
  op __ : Procs Procs -> Procs [assoc comm id: mt] .    *** union
  op __ : WProcs WProcs -> WProcs [assoc comm id: mt] . *** union
  op _|_|_ : Nat Nat? Procs -> State .
  vars n m i j k : Nat . var x? : Nat? . var PS : Procs . var WPS : WProcs .
  rl [new]:  m | n | PS => s(m) | n | < wait,m > PS [narrowing] .
  rl [enter]: m | n | < wait,n > PS => m | [n] | < crit,n > PS [narrowing] .
  rl [leave]: m | [n] | < crit,n > PS => m | s(n) | PS [narrowing] .
endm
```

States have the form "`m | x? | PS`" with `m` the ticket counter, `x?` the counter to access the critical section, and `PS` a multiset of processes. `BAKERY` is infinite-state because of `[new]`. When a waiting process `n` enters the critical section, the second counter `n` is locked as `[n]`; and it is unlocked and incremented when `n` leaves.

Lamport's Bakery Protocol (II)

The key invariant is mutual exclusion. Its complement is specified by the term `i | x? | < crit, j > < crit, k > PS`.

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Maude> {fold} vu-narrow
        m | n | WPS =>* i | x? | < crit, j > < crit, k > PS .

No solution.
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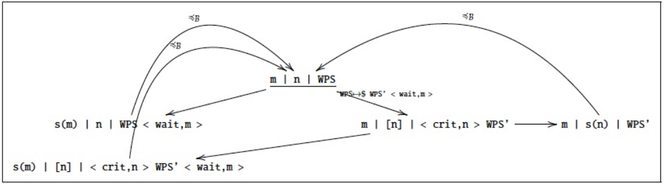
No solution.
```

We can visualize the dramatic state space reduction obtained by folding an infinite tree of symbolic states into a finite graph with only four states in the figure below.

Lamport's Bakery Protocol (III)



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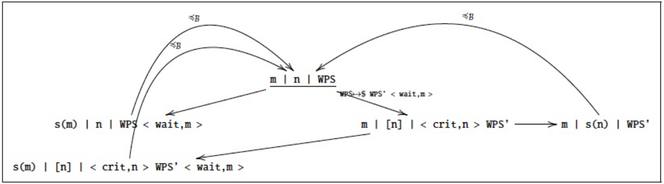

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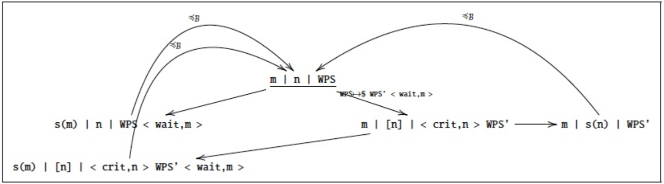

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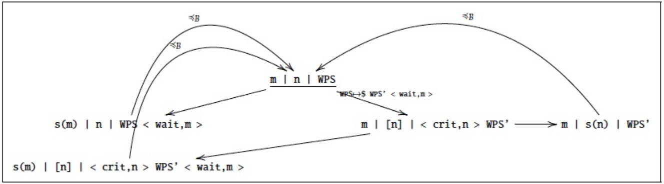

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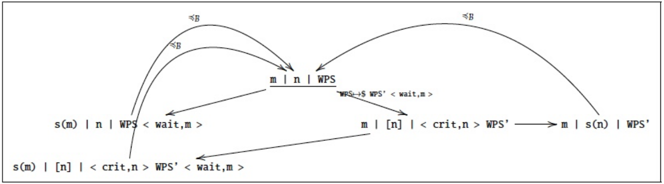

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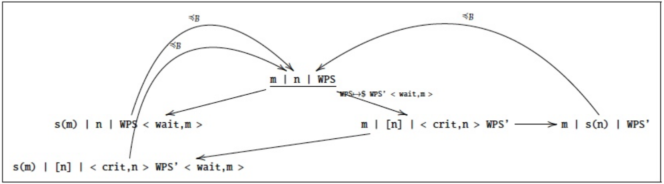

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The reason is that, if we start with a quite specific initial state, the subsequent symbolic states will be even more specific. This is what the word "narrowing" means. The more specific a state is, the less it can generalize other symbolic states by folding them. Worse of all are ground initial states like `0 | 0 | mt`, which turn narrowing search into rewriting search and make generalization impossible.

Initial States as Disjunctions of Patterns

Since the more general the initial state, the better, an important way of achieving such generality, so as to increase the chances that variant narrowing search terminates, is to allow initial states to be a disjunction of patterns. Indeed, Maude 3.5 allows narrowing search commands of the form:

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This means that we can try to solve existential reachability formulas of the form $\exists \vec{x}, \vec{y},\ u_1 \vee \ldots \vee u_n \rightarrow^* v_1 \vee \ldots \vee v_m$ by means of m `{fold} vu-narrow` commands of the above form.

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We shall further explore the advantages of this greater generality for specifying initial states in Lecture 24.