Appendix to Lecture 25

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Theorem. Let $f : \mathcal{A} \to \mathcal{B}$ be a *simulation* map, then for any $U, V \subseteq A$, $\mathcal{A} \models \exists U \to^* V$ implies $\mathcal{B} \models \exists f(U) \to^* f(V)$. Equivalently, $\mathcal{B} \models \forall f(U) \twoheadrightarrow^* f(V)$ implies $\mathcal{A} \models \forall U \twoheadrightarrow^* V$.

Proof: $\mathcal{A} \models \exists U \to^* V$ just means that $\exists a \in U,\ \exists a' \in V,\ a \to^*_{\mathcal{A}} a'$. But, since f is a simulation, this forces $f(a) \to^*_{\mathcal{B}} f(a')$, i.e., $\mathcal{B} \models \exists f(U) \to^* f(V)$. □

Theorem. Let $f : \mathcal{A} \to \mathcal{B}$ be a *bisimulation* map, then for any $U, V \subseteq A$, $\mathcal{A} \models \exists U \to^* V$ iff $\mathcal{B} \models \exists f(U) \to^* f(V)$. Equivalently, $\mathcal{A} \models \forall U \twoheadrightarrow^* V$ iff $\mathcal{B} \models \forall f(U) \twoheadrightarrow^* f(V)$.

Proof: Taking into account the previous theorem, we just need to prove that $\mathcal{B} \models \exists f(U) \to^* f(V)$ implies $\mathcal{A} \models \exists U \to^* V$. But $\mathcal{B} \models \exists f(U) \to^* f(V)$ means that $\exists a \in U,\ \exists a' \in V,\ f(a) \to^*_{\mathcal{B}} f(a')$, and f being a bisimulation then forces $a \to^*_{\mathcal{A}} a'$, giving us $\mathcal{A} \models \exists U \to^* V$, as desired. □

Theorem. Let $\mathcal{R} = (\Sigma, E \cup B, R)$ be a topmost rewrite theory such that $G = E \cup B$ is FVP, and $G' = E' \cup B'$ is such that $E \cup E' \cup B \cup B'$ is FVP modulo $B \cup B'$. $\mathcal{R}/G'$ defines a bisimilar equational abstraction of $\mathcal{R}$ if for each $(u^i_0 = u^i_1) \in G'$, $1 \leqslant i \leqslant p$, and $(t^j_0 \to t^j_1) \in R$, $1 \leqslant j \leqslant q$, and each $\sigma \in \mathit{Unif}_G(t^j_{b'} = u^i_b)$, $0 \leqslant b \leqslant 1$, $0 \leqslant b' \leqslant 1$, there exists a θ such that $u^i_{b' \oplus 1}\sigma =_G t^j_b\theta \ \wedge\ t^j_{b\oplus 1}\theta =_G t^j_{b\oplus 1}\sigma$, where $\oplus$ denotes exclusive or.

Proof: It will be enough to show that for all j, $1 \leqslant j \leqslant q$, and for any $R/G \cup G'$ rewrite of the form $w =_{G\cup G'} t^j_0\rho \to t^j_1\rho =_{G\cup G'} w'$ there is a R/G rewrite of the form $w =_G t^j_0\rho' \to t^j_1\rho' =_G w'$. The proof is by contradiction. Suppose not. Then, we can choose a specific k, $1 \leqslant k \leqslant q$, and an $R/G \cup G'$ rewrite $w =_{G\cup G'} t^k_0\rho \to t^k_1\rho =_{G\cup G'} w'$ for which no R/G rewrite of the form $w =_G t^k_0\rho' \to t^k_i\rho' =_G w'$ exists and, furthermore, if n_l are the G' equality steps in $w =_{G\cup G'} t^k_0\rho$ and n_r are the G' equality steps in $t^k_1\rho =_{G\cup G'} w'$, then $n_l + n_r$ is *smallest possible* so that no R/G rewrite of the form $w =_G t^k_0\rho' \to t^k_1\rho' =_G w'$ exists. Without loss of generality we may assume that $\mathit{vars}(G') \cap \mathit{vars}(R) = \varnothing$ and that $n_l > 0$ (the case $n_r > 0$ is entirely analogous). Therefore, the proof $w =_{G\cup G'} t^k_0$ can be decomposed as a proof of the form $w =_{G\cup G'} u^{k'}_{b\oplus 1}\tau = u^{k'}_b\tau =_G t^k_0\rho$ for some k', $1 \leqslant k' \leqslant p$, and τ, with $w =_{G\cup G'} u^{k'}_{b\oplus 1}\tau$ involving $n_l - 1$ G' equality steps. But then $\tau \uplus \rho$ is a G-unifier of the equation $u^{k'}_b = t^k_0$. Therefore, there exists a $\sigma \in \mathit{Unif}_G(t^k_0 = u^{k'}_b)$ and a γ such that $\sigma\gamma =_G (\tau \uplus \rho)$, and by the theorem's assumptions there exist a θ such that $u^{k'}_{b'\oplus 1}\sigma =_G t^k_0\theta \ \wedge\ t^k_1\theta =_G t^k_1\sigma$. But this gives as a $R/G \cup G'$ rewrite of the form:

$$w =_{G\cup G'} u^{k'}_{b\oplus 1}\tau =_G u^{k'}_{b\oplus 1}\sigma\gamma =_G t^k_0\theta\gamma \to t^k_1\theta\gamma =_G t^k_1\sigma\gamma =_G t^k_1\rho =_{G\cup G'} w'$$

with $(n_l + n_r) - 1$ G' equality steps, contradicting the minimality of $n_l + n_r$ for which no R/G rewrite of the form $w =_G t^k_0\rho' \to t^k_1\rho' =_G w'$ exists. □