Program Verification: Lecture 24

José Meseguer
University of Illinois at Urbana-Champaign

Symbolic Model Checking Modulo an FVP Theory $E \cup B$

- In Lecture 23, narrowing-based symbolic model checking was extended from topmost rewrite theories $\mathcal{R} = (\Sigma, B, R)$ to topmost theories $\mathcal{R} = (\Sigma, E \cup B, R)$, with $E \cup B$ FVP.
- The extension was very smooth:
  - Instead or narrowing with R modulo axioms B by performing B-unification, one narrows with R modulo axioms $E \cup B$ by performing $E \cup B$-variant unification.
  - To try to make the narrowing symbolic search space finite, instead of folding symbolic states that are instances modulo axioms B of more general states, we fold them into more general states symbolic states of which they are instances modulo $E \cup B$.
  - In both cases, the `fvu-narrow` command in Maude supports symbolic model checking with narrowing.

In this lecture I will: (1) illustrate this kind of symbolic reachability analysis with folding modulo an FVP theory $E \cup B$ with two infinite-state system examples, and (2) will show how the folding narrowing graph $FG_{\mathcal{R}}(u)$ from a symbolic initial state u faithfully characterizes the satisfaction (resp. violation) of invariants in $\mathcal{R}$.

VENDING-MACHINE

The following vending machine allows buying cakes or cookies with either dollars or quarters thanks to the FVP equation: $q\ q\ q\ q = \$$.

```
mod VENDING-MACHINE is
  sorts Coin Item Marking Money State .
  subsort Coin < Money .
  op empty :  -> Money .
  op __ :  Money Money -> Money [assoc comm id:  empty] .
  subsort Money Item < Marking .
  op __ :  Marking Marking -> Marking [assoc comm id:  empty] .
  op <_> :  Marking -> State .
  ops $ q :  -> Coin .
  ops cookie cake :  -> Item .
  var M : Marking .
  rl [add-$] :  < M > => < M $ > .
  rl [add-q] :  < M > => < M q > .
  rl [buy-ca] :  < M $ > => < M cake > .
  rl [buy-co] :  < M $ > => < M cookie q > .
  eq [change]:  q q q q = $ [variant] .
endm
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$$\begin{tikzcd} < \$\ \$ > \arrow[r] \arrow[dr] & < \$\ \$\ \$ > \arrow[r] & < \$\ \$\ \$\ \$ > \arrow[r] & \infty \\ & < \$\ \$\ q > \arrow[r] & < \$\ \$\ q\ q > \arrow[r] & \infty \end{tikzcd}$$

(one initial state - infinite space)

Narrowing-Based Symbolic Model Checking

- We can consider, for example, the most general symbolic initial state possible in `VENDING-MACHINE`, namely, < `M` > and its symbolic transitions by the `[buy-ca]` rule.
- The vertical lines in the figure below describe the narrowing steps and unifiers for the narrowing path:

$$\langle M \rangle \rightsquigarrow \langle \mathit{cake}\ M' \rangle \rightsquigarrow \langle \mathit{cake}\ \mathit{cake}\ M'' \rangle \rightsquigarrow \ldots$$

$$\begin{tikzcd}
\texttt{< M >} \arrow[d, "\texttt{\{M}\mapsto\texttt{\$ M'\}}"'] & \texttt{< cake M' >} \arrow[d, "\texttt{\{M'}\mapsto\texttt{\$ M''\}}"'] & \texttt{< cake cake M'' >} \arrow[d] \\
\texttt{< cake M' >} \arrow[ur, bend left] & \texttt{< cake cake M'' >} \arrow[ur, bend left] & \infty
\end{tikzcd}$$

(Infinite Symbolic Search Space)

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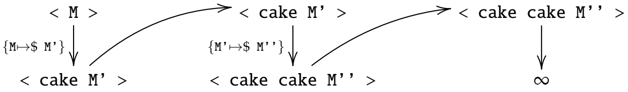

(Infinite Symbolic Search Space)

Folding Infinite Symbolic State Spaces into a Finite Graph

- Narrowing-Based Symbolic Model Checking can model check infinite state systems by representing infinite sets of states by terms with variables.
- The symbolic state space (narrowing tree) can still be infinite; but its states can be over-approximated by folding in the folding graph, which sometimes can be finite.
- The transition system of the folding graph is an abstraction (it identifies many symbolic states) that over-approximates the states and transitions of the narrowing tree.

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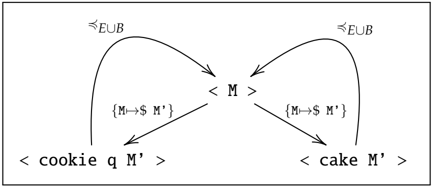

Narrowing + folding relation ⇒ (symbolic initial states and (hopefully) finite state space)
(instantiation relation $\preccurlyeq_{E \cup B}$)

$E \cup B$-Unification Command in Maude

Maude provides a $(E \cup B)$-unification command for any equational theory $(\Sigma, E \cup B)$ that is convergent modulo B. The complete set of $(E \cup B)$-unifiers will always be finite if $E \cup B$ is FVP.

$$\texttt{variant unify } [\ \texttt{in } \langle \textit{ModId} \rangle\ \texttt{:}\ ]\ \langle \textit{Term1} \rangle\ \texttt{=?}\ \langle \textit{Term2} \rangle\ \texttt{.}$$

- ModId is the name of the module
- A complete set of $E \cup B$-unifiers are returned.
- Folding variant narrowing is used internally to compute $E \cup B$-unifiers.

$(E \cup B)$-Unification Command in Maude (II)

```
Maude> (variant unify in NARROWING-VENDING-MACHINE :
          < q q X:Marking > =? < $ Y:Marking > .)
Solution 1
X:Marking --> q q Y:Marking
Solution 2
X:Marking --> $ #12:Marking ; Y:Marking --> q q #12:Marking
```

Bakery Algorithm: Transition System

Token to give ;	Token serving ;	Set of Processes
Nat	Nat	[{ idle, wait(Nat), crit(Nat) }]

$$
\begin{array}{llll}
\texttt{rl}\ N ; M ; [\mathrm{idle}]\ PS & \Rightarrow (s\,N) ; & M & ; [\mathrm{wait}(N)]\ PS\ . \\
\texttt{rl}\ N ; M ; [\mathrm{wait}(M)]\ PS & \Rightarrow \ N ; & M & ; [\mathrm{crit}(M)]\ PS\ . \\
\texttt{rl}\ N ; M ; [\mathrm{crit}(M)]\ PS & \Rightarrow \ N ; & (s\,M) & ; [\mathrm{idle}]\ PS\ .
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\end{array}
$$

$$
\begin{tikzcd}
\texttt{0 ; 0 ; [idle]} \arrow[d] & \texttt{s ; s ; [idle]} \arrow[d] & \texttt{s s ; s s ; [idle]} \arrow[d] \\
\texttt{s ; 0 ; [wait(0)]} \arrow[d] & \texttt{s ; s ; [wait(s)]} \arrow[d] & \texttt{s s s ; s s ; [wait(s s)]} \arrow[d] \\
\texttt{s ; 0 ; [crit(s)]} \arrow[uur] & \texttt{s s ; s ; [crit(s)]} \arrow[uur] & \infty
\end{tikzcd}
$$

(Transition System: one initial state - infinite space)

Bakery Algorithm: Symbolic Transition System

$$
\begin{tikzcd}
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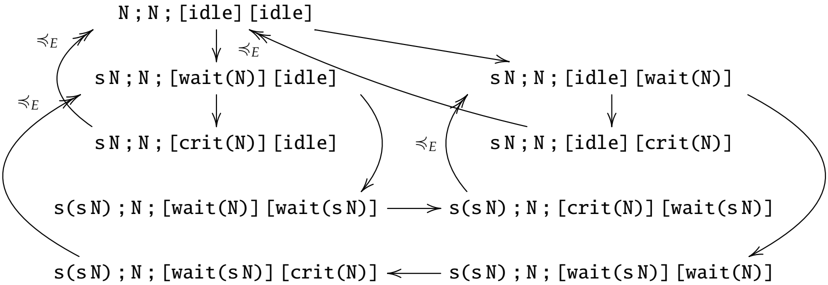
(Transition System: one initial state - infinite state space)

$$
\begin{tikzcd}
\texttt{N ; N ; [idle]} \arrow[d] & \texttt{s N ; s N ; [idle]} \arrow[d] & \texttt{s s N ; s s N ; [idle]} \arrow[d] \\
\texttt{s N ; N ; [wait(N)]} \arrow[d] & \texttt{s s N ; s N ; [wait(s N)]} \arrow[d] & \texttt{s s s N ; s s N ; [wait(s s N)]} \arrow[d] \\
\texttt{s N ; N ; [crit(N)]} \arrow[uur] & \texttt{s s N ; s N ; [crit(s N)]} \arrow[uur] & \cdots
\end{tikzcd}
$$

(Symbolic Transition System: infinite initial state set - infinite state space)

Bakery Algorithm: Folding the Symbolic Transition System



(Folding Symbolic Transition System : infinite initial state set - finite state space)

The Faithfulness of Folding Symbolic Transition Systems

Suppose that we wish to verify an invariant I for a topmost $\mathcal{R}$ using the folding graph $FG_{\mathcal{R}}(u)$ generated by a symbolic initial state u. Since $FG_{\mathcal{R}}(u)$ over-approximates the narrowing tree from u, if no violation of invariant I (i.e., an instance of u reaching its complement) is found exploring $FG_{\mathcal{R}}(u)$, a fortiori no such violation can be found in the narrowing tree. But by the Completeness of Narrowing Search Theorem (Lecture 23, pg. 8), this means that I holds for all ground instances of u.

But what happens if we find a counterexample, that is, a path from u in $FG_{\mathcal{R}}(u)$ violating I? Does it mean that invariant I is violated for some ground instance of u? Or could such a path be a spurious counterexample not corresponding to any real violation of I?

We shall call $FG_{\mathcal{R}}(u)$ a faithful abstraction of $\mathcal{R}$ from the set of initial states symbolically specified by u iff $FG_{\mathcal{R}}(u)$ has no spurious counterexamples for any pattern-specified invariant I. To show that $FG_{\mathcal{R}}(u)$ is faithful, we need to look at it more carefully.

The Folding Narrowing Graph $FNG_{\mathcal{R}}(u)$

Given a topmost $\mathcal{R} = (\Sigma, E \cup B, R)$ with $E \cup B$ FVP, and a symbolic initial state u, the folding narrowing graph $FNG_{\mathcal{R}}(u)$ is generated in a breadth first manner by paths of increasing length from u as follows:

- $u \leadsto_{R,(E\cup B)} u$ is the only path at depth 0.
- The paths of lenght $n+1$ (and the depth of their ending nodes) are:
  - either narrowing paths $u \leadsto^{n}_{R,(E\cup B)} v_n \leadsto_{R,(E\cup B)} v$ such that (i) $u \leadsto^{n}_{R,(E\cup B)} v_n$ in $FNG_{\mathcal{R}}(u)$, (so v has narrowing depth n+1 in u's narrowing tree), and it is not the case that (ii): either exists a narrowing path $u \leadsto^{k}_{R,(E\cup B)} w$, $k \leq n$, in $FNG_{\mathcal{R}}(u)$, or a different narrowing path $u \leadsto^{n}_{R,(E\cup B)} w_n \leadsto_{R,(E\cup B)} w$ with $u \leadsto^{n}_{R,(E\cup B)} w_n$ in $FNG_{\mathcal{R}}(u)$, such that $v \preccurlyeq_{E\cup B} w$ (read, v is an instance modulo $E \cup B$ of w), where,

$$v \preccurlyeq_{E\cup B} w \quad \Leftrightarrow_{def} \quad \exists\gamma \;\; s.t. \;\; w\gamma =_{E\cup B} v;$$

  - otherwise, they are paths of the form $u \leadsto^{n}_{R,(E\cup B)} v_n \preccurlyeq_{E\cup B} w$ associated to a narrowing path $u \leadsto^{n}_{R,(E\cup B)} v_n \leadsto_{R,(E\cup B)} v$ s.t. (i)–(ii) above hold with $v \preccurlyeq_{E\cup B} w$. Therefore, w has narrowing depth $d \leq n+1$ in u's narrowing tree.

Faithfulness of $FNG_{\mathcal{R}}(u)$ (proofs in Appendix)

Theorem

(Over-Approximation Theorem). Given a topmost $\mathcal{R} = (\Sigma, E \cup B, R)$ *with* $E \cup B$ *FVP and a symbolic initial state* u*, for every narrowing path from* u*,* $u \rightsquigarrow^*_{R,(E \cup B)} v$ *there is a node* w *in the folding narrowing path of* u $FNG_{\mathcal{R}}(u)$ *such that* $v \preccurlyeq_{E \cup B} w$*.*

Theorem

(Faithfulness Theorem). For $\mathcal{R} = (\Sigma, E \cup B, R)$ *and* u *as above,* $FNG_{\mathcal{R}}(u)$ *is a faithful over-approximation of the narrowing tree of* u *in the sense that for any set of states of* $\mathcal{R}$ *described by a pattern term* p*, an instance of* p *can be reached by a narrowing path* $u \rightsquigarrow^*_{R,(E \cup B)} v$ *such that* $Unif_{E \cup B}(v = p) \neq \emptyset$ *iff there is a node* w *in* $FNG_{\mathcal{R}}(u)$ *such that* $Unif_{E \cup B}(w = p) \neq \emptyset$*.*

In particular, if p *is the negation of an invariant, any counterexample found in* $FNG_{\mathcal{R}}(u)$ *is a true counterexample and therefore proves the invariant's violation (i.e.,* $FNG_{\mathcal{R}}(u)$ *has no spurious counterexamples).*